SUMMARY

Harmonic maps are the solutions of a natural variational problem in Differential Geometry. This thesis is concerned with questions of existence, classification and special properties of harmonic maps.

1. Existence:

Variational arguments are used to establish the existence of harmonic maps of finite energy from non-compact manifolds when either

(a) the target manifold is compact and satisfies certain geometrical conditions, or

(b) the domain is two-dimensional and the target satisfies certain growth conditions.

Further, infinite-dimensional differentiable structures are exhibited for certain spaces of maps that arise naturally in this context.

2. Classification:

The twistorial methods of Eells-Salamon and Rawnsley are exploited to classify strongly conformal harmonic maps of a Riemann surface into a Grassmannian by holomorphic maps of the surface into a flag manifold equipped with a special non-integrable almost complex structure.

Similar ideas are used to classify isotropic harmonic maps of a Riemann surface into a space form by f-holomorphic maps into bundles of f-structures over the space form.

In this context, we also examine the relevant properties of f-structures and f-holomorphic maps and, in particular, show the existence of a homotopy invariant for maps of cosymplectic manifolds into f-Kähler manifolds generalising that of Lichnerowicz.

3. Properties:

A characterisation in terms of harmonic maps of those maps between Riemannian manifolds that commute with the co-differential is given.

Unique continuation properties of harmonic maps are considered and in the case of two-dimensional domains, proved by use of holomorphic differentials. In particular, we establish unique continuation of isotropy for branched minimal surfaces in a space form.

NON-LINEAR FUNCTIONAL ANALYSIS AND HARMONIC MAPS

by

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INTRODUCTION

1. PRELIMINARIES

Let us begin with a rapid view of some notions in Riemannian geometry to fix ideas and notation.

A. *Structures and operators on a vector bundle (see [45]).*

Let (M,g) be a Riemannian manifold, that is a smooth manifold with a smooth choice of inner product on each tangent space and let $\pi : E \to M$ be a smooth vector bundle.

A *connection* on E is a linear operator $\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$ such that:

$$\nabla f\sigma = df \otimes \sigma + f\nabla\sigma$$

for $f \in C^\infty(M)$, $\sigma \in C^\infty(E)$. If $X \in TM$, we denote $\nabla\sigma(X)$ by $\nabla_X\sigma$.

Now, if E and F are vector bundles over M with connections ∇^E, ∇^F respectively, we can build connections on the various dual, tensor and product bundles of E and F by demanding that each such connection is a derivation commuting with contractions, e.g:

$$\nabla(\sigma_1 \otimes \sigma_2) = \nabla^E\sigma_1 \otimes \sigma_2 + \sigma_1 \otimes \nabla^F\sigma_2 \ ,$$

$$(\nabla\sigma^*)(\xi) = d(\sigma^*.\xi) - \sigma^*(\nabla^E\xi) \ ,$$

for $\sigma_1, \xi \in C^\infty(E)$, $\sigma^* \in C^\infty(E^*)$, $\sigma_2 \in C^\infty(F)$.

The *curvature*, R, of a connection ∇ is a section of $\Lambda^2T^*M \otimes \mathrm{End}(E)$ given by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \ .$$

Now let ∇ be a connection on TM, the tangent bundle of M. The *torsion* of ∇ is a section of $\Lambda^2 T^*M \otimes TM$, denoted by T, given by

$$T(X,Y) = -\nabla_X Y + \nabla_Y X + [X,Y] .$$

As is well-known, there is a unique connection on TM which is torsion-free and with respect to which the metric is parallel i.e.

$$\nabla g \equiv 0$$

$$\nabla_X Y - \nabla_Y X = [X,Y] .$$

This connection is called the *Levi-Civita connection* for (M,g).

Now let $x \in M$ and P an oriented 2-plane in $T_x M$ with orthonormal basis X,Y. The *sectional curvature* of P, denoted Riem(P), is given by

$$\mathrm{Riem}(P) = g(R(X,Y)X,Y)$$

where R is the curvature of the Levi-Civita connection. This definition is independent of the choice of ~~oriented~~ orthonormal basis.

We say that (M,g) has constant (positive, etc.) sectional curvatures if Riem(P) is constant (positive, etc.) for all 2-planes P in TM.

An *almost complex structure* on M is a section of End(TM) such that

$$J^2 = -\mathrm{Id},$$

on each fibre. A manifold together with an almost complex structure is called an *almost complex manifold*.

An almost complex manifold is said to have *integrable* almost complex structure or to be a *complex manifold* if M admits an atlas of holomorphic charts and in such a chart, J is just multiplication by $i = \sqrt{-1}$. By the theorem of Newlander-Nirenberg, this is equivalent to the vanishing of the Nijenhuis tensor, N^J, given by

$$N^J(X,Y) = 2\{[JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]\} .$$

An (almost) complex manifold, (M,g,J), is *(almost) Hermitian* if J is an isometry on each fibre, in which case we define the *Kähler form*, ω, by

$$\omega(X,Y) = g(X,JY),$$

and (M,g,J) is *(almost) Kähler* if ω is closed.

An almost Hermitian manifold is Kähler (i.e. integrable and almost Kähler) if and only if J is parallel with respect to the Levi-Civita connection.

Example Let (M,g) be two-dimensional and oriented. Define an almost complex structure J on M by setting J to be rotation by $\frac{\pi}{2}$ in a positive sense on each fibre. Then (M,g,J) is almost Hermitian and it follows from the dimension of M that both N^J and $d\omega$ vanish so that (M,g,J) is a Kähler manifold.

B. *Harmonic Maps*

Let $\phi : (M,g) \to (N,h)$ be a smooth map of Riemannian manifolds. Its derivative, $d\phi$, is a section of $T^*M \otimes \phi^{-1}TN$, where $\phi^{-1}TN \to M$ is the pull-back of TN by ϕ, i.e. the bundle with fibre at $x \in M$ given by $T_{\phi(x)}N$.

The *second fundamental form*, β_ϕ, of ϕ is the section of $\otimes^2 T^*M \otimes \phi^{-1}TN$ given by

$$\beta_\phi(X,Y) = \nabla d\phi(X,Y),$$

for $X,Y \in TM$, where $\hat{\nabla}$ is the connection on $T^*M \otimes \phi^{-1}TN$ induced from the

Levi-Civita connections on M and N.

It follows from the fact that Levi-Civita connections are torsion-free that the second fundamental form is symmetric:

$$\beta_\phi(X,Y) = \beta_\phi(Y,X).$$

A map with vanishing second fundamental form is said to be *totally geodesic*.

The *energy* of $\phi : (M,g) \to (N,h)$, denoted $E(\phi)$, is given by

$$E(\phi) = \tfrac{1}{2}\int_M \langle d\phi, d\phi\rangle dv_g,$$

where the inner product is the Hilbert-Schmidt inner product on $T^*M \otimes \phi^{-1}TN$ and dv_g is the volume element of (M,g).

A C^2 map $\phi : (M,g) \to (N,h)$ is said to be *harmonic* if ϕ extremises the energy with respect to all compactly supported variations in $C^\infty(M,N)$.

The Euler-Lagrange operator of the energy functional, the *tension field*, is denoted by τ and, at ϕ, is a section of $\phi^{-1}TN$ given by

$$\tau_\phi = \text{Trace}\,\beta_\phi = d^* d\phi ,$$

where d^* is the co-differential on $\phi^{-1}TN$-valued 1-forms. Thus, since $dd\phi$ always vanishes, we see by integration by parts, that ϕ is harmonic if and only if $d\phi$ is a harmonic $\phi^{-1}TN$-valued 1-form when M is compact.

In local co-ordinates $\{x_i\}_i$ on M and $\{y_\alpha\}_\alpha$ on N;

$$\tau_\phi^\alpha = g^{ij}\left(\frac{\partial\phi^\alpha}{\partial x_i \partial x_j} - {}^M\Gamma_{ij}^k \frac{\partial\phi^\alpha}{\partial x_k} + {}^N\Gamma_{\beta\gamma}^\alpha \frac{\partial\phi^\beta}{\partial x_i}\frac{\partial\phi^\gamma}{\partial x_j}\right).$$

where ${}^M\Gamma_{ij}^k$, ${}^N\Gamma_{\beta\gamma}^\alpha$ are the Christoffel symbols of M and N respectively.

From this last equation, we see that harmonic maps are locally solutions of a system of second-order semi-linear elliptic partial differential equations. Thus, elliptic regularity theory implies that harmonic maps of smooth Riemannian manifolds are automatically smooth, [52].

Harmonic maps occur naturally in many different areas of Differential Geometry, for example:

(i) For $\dim M = 1$, harmonic maps are precisely geodesics parametrised by arc-length.

(ii) For $N = \mathbb{R}^n$, the tension field is linear and, in fact, is just the Laplace-Beltrami operator on M, thus harmonic maps are just harmonic functions.

(iii) Minimal immersions are precisely the harmonic isometric immersions.

(iv) Holomorphic maps between Kähler manifolds are harmonic [~~50~~ 27].

(v) It is obvious from the above that totally geodesic maps are harmonic.

The best references for harmonic maps are the survey articles of Eells-Lemaire [22,23].

2. HARMONIC MAPS AND DIFFERENTIAL GEOMETRY

As solutions of a system of elliptic Euler-Lagrange equations, harmonic maps enjoy a number of special properties. For instance, they possess the unique continuation property [62], are subject to a maximum principle [62] and reflection principle [74] and under suitable hypotheses on curvature, Liouville-type theorems may be proved [66].

However, in order to apply these results, we need a good supply of harmonic maps. In particular, given Riemannian manifolds M and N, are there harmonic maps in every homotopy class (component) of $C^\infty(M,N)$? In general the answer is no: Eells and Wood have shown that there is no harmonic map of degree one of a 2-torus into S^2,[28]. However, under suitable geometric conditions on N, we can use a variety of techniques to provide an affirmative answer:

(a) Heat-flow: In 1964, Eells and Sampson, [27], got the theory of harmonic maps off the ground by showing that, in case M is compact and without boundary and N has non-positive sectional curvatures and is compact, deformation along the heat-flow associated with the tension field produces an energy minimising harmonic map in every component of $C^\infty(M,N)$. This result was later extended to compact domains with boundary by Hamilton [38].

(b) Morse Theory: Uhlenbeck, [72], provided a proof of the Eells-Sampson theorem by applying Morse theory to a perturbation of the energy functional on a suitable manifold of maps.

(c) Direct Method of the Calculus of Variations: Hildebrandt *et al.* [41], extended the Eells-Sampson theorem to target manifolds with positive curvature under condition that the harmonic map has image contained in a sufficiently small geodesic ball (the radius of which is bounded above in terms of $\frac{1}{\sqrt{K}}$ where K is an upper bound for the sectional curvatures on the ball). Their proof proceeded by extracting a weakly convergent subsequence of an energy minimising sequence of maps in $L_1^2(M,N)$ and proving regularity of the energy minimising limit map.

The recent regularity theory of Schoen and Uhlenbeck, [64], allows us to interpret the various curvature hypotheses as follows: according to Schoen-Uhlenbeck, the obstruction to regularity of an energy-minimising harmonic map into N is the existence of non-constant harmonic maps of spheres into N. Now, the curvature hypotheses of (a),(b) or (c) above imply the existence of a convex function on the universal cover of N which, in turn, implies the non-existence of non-constant harmonic spheres in N (see [20]). Thus the Schoen-Uhlenbeck theory together with a direct variational argument unifies and extends the results above.

In case $\dim M = 2$, the theory of harmonic maps has special features, one of which is that the regularity theory for energy-minimising maps (here due to Morrey [51,52]) requires no hypotheses on N. Thus Lemaire [48] was able to prove the existence of a harmonic map in each component of $C^{\infty}(M,N)$ where M is a compact surface and N is any compact Riemannian manifold with vanishing second homotopy group (see also Sacks-Uhlenbeck [60] and Schoen-Yau [68]).

However, many target manifolds of interest do not satisfy the curvature or topological restrictions mentioned above; for instance, Riemannian symmetric spaces of compact type. Further, in such cases, the variational approach does not seem appropriate; for example, the component of the identity map in $C^{\infty}(S^n,S^n)$, $n \geq 3$, contains maps of arbitrarily small energy and so cannot contain an energy-minimising map (see [23]).

Thus we must seek new methods for finding harmonic maps. In case $\dim M = 2$ we may exploit the holomorphic structure of M and holomorphic differentials on M associated to harmonic maps. Using such ideas, Calabi exhibited a (constructive) 2:1 correspondence between full totally isotropic holomorphic maps of a Riemann surface M into $\mathbb{C}P^{2n}$ and full isotropic harmonic maps of M into S^{2n} thus reducing the existence problem for harmonic maps to Algebraic Geometry, [13,14]. This approach was taken up by Eells and Wood who treated isotropic harmonic maps of surfaces into complex projective spaces [29] and Erdem and Wood who treated maps into complex Grassmannians [31].

These results may be thought of as a kind of twistor correspondence, associating isotropic harmonic maps into N with holomorphic maps into a bundle of almost complex structures over N : a twistor space. The celebrated fibration of $\mathbb{C}P^3 \to S^4$ considered by Atiyah *et al.* [6] is an example of this set-up.

Recently, this approach has been greatly extended in scope in two ways: firstly, Eells and Salamon, [25], have introduced a non-integrable almost complex structure on twistor bundles in terms of which they have classified all conformal harmonic maps (branched minimal immersions) of Riemann surfaces into a 4-manifold. Secondly, Rawnsley [59] has introduced bundles of f-structures (in the sense of Yano [77]) and used these as twistor spaces. These topics will be taken up in detail in chapters 5-7 of this work.

We conclude this section by briefly mentioning some applications of harmonic maps:

1. Relevant to the above twistor theory, there is the theorem of Ruh and Vilms [58] which shows that an isometric immersion into $\mathbb{R}^n$ has constant mean curvature if and only if its Gauss map is harmonic as a map into a suitable Grassmannian.

2. Coupling the existence of harmonic representatives of homotopy classes with Liouville-type theorems leads to results such as that of Schoen and Yau [66]:

Let (M,g) be complete with positive semi-definite Ricci curvature and (N,h) compact with non-positive sectional curvatures, then any map with finite energy from M to N is null-homotopic.

3. Under suitable conditions, harmonic maps of Kähler manifolds can be shown to be holomorphic (see Eells-Wood [28], Siu [69]). This kind of argument was a key step in Siu and Yau's proof of Fraenkel's conjecture, [70].

3. THE CONTENTS OF THIS THESIS

This thesis is mainly concerned with the existence questions for harmonic maps considered in §2.

In Chapter 1, motivated by the regularity theorems of Schoen-Uhlenbeck and Giaquinta-Giusti [32,33], we prove the existence of a smooth infinite-dimensional differentiable structure on the space of essentially bounded L_1^2 maps between two Riemannian manifolds and exhibit a natural Finsler structure thereon. Our methods are valid for the space of continuous L_1^2 maps which is relevant to the theory of harmonic maps since weakly harmonic maps in this space are smooth by a theorem of Hildebrandt *et al.* [42].

In the hope that some sort of critical point theory for the energy functional on these manifolds might become available, we examine various aspects of the theory of harmonic maps in this setting. However, as Lemaire has pointed out, the non-existence results considered in §2 show that we cannot expect Palais-Smale condition 'C' for the energy functional in general.

In Chapter 2, we turn to maps of non-compact manifolds. For M a complete non-compact Riemannian manifold and N a compact Riemannian manifold satisfying the currently optimal regularity conditions of Schoen-Uhlenbeck (i.e. N admits no non-constant harmonic spheres), we prove the existence of an energy-minimising harmonic map in every component of $C^\infty(M,N)$ which contains a map of finite energy. This extends results of Schoen and Yau [66] (N has non-positive curvature) and

Lemaire [49] ($\dim M = 2$) and brings the case of non-compact M into line with what is known for compact M (see §2). The main technical result in the proof is the production of a homomorphism on fundamental groups induced by a discontinuous L_1^2 map. This result (due to Schoen-Yau [68] for $\dim M = 2$) has been claimed by Schoen [63] but there is no proof of it in the literature. Thus our arguments provide the 'missing link' in a variational existence theory of harmonic maps even for compact M.

Lastly, for $\dim M = 2$, our methods allow us to treat the case of N non-compact but satisfying a growth condition of Lemaire.

In Chapters 3 and 4 we examine some properties of harmonic maps. In Chapter 3, we characterise those maps of Riemannian manifolds which commute with the co-differential on vector-bundle-valued forms, and provide a counter-example to the previous incorrect characterisation of Watson, [73].

In Chapter 4, we consider unique continuation properties of harmonic maps and harmonic vector-bundle-valued forms (such as Yang-Mills fields). The main analytic tool is the unique continuation theorem of Aronszajn, Kryzwicki and Szarski, [4], which we extend to vector-bundle-valued forms. However, in case that the domain is a Riemann surface, such properties may be proved directly by exploiting the holomorphic differentials associated with harmonic maps in this case. In particular, we define a set of differentials linked to the isotropy of harmonic maps that enable us to prove unique continuation of isotropy for branched minimal immersions of Riemann surfaces into space forms. These differentials are also known to Wood, [75].

The last three chapters are concerned with the twistor programme developed by Eells-Salamon and Rawnsley mentioned in §2.

Motivated by Rawnsley's bundles of f-structures, in Chapter 5 we consider maps between manifolds with f-structures and conditions on such f-structures that ensure the harmonicity of f-holomorphic maps. There are two such conditions, one due to Rawnsley [59]; condition 'A', which in some sense complement each other; one being appropriate for twistor bundles and the other for such manifolds as pseudo-convex hypersurfaces in $\mathbb{C}^n$ with their natural Cauchy-Riemann structures.

In Chapter 6, we consider bundles of f-structures over Grassmannians that are specially adapted to the homogeneous structure. These smaller twistor bundles admit very well-behaved f-structures and we are able, for instance, to obtain a bijective correspondence between harmonic maps of a surface, M, into a complex Grassmannian satisfying a strong conformality condition and certain maps of M into a flag manifold which are holomorphic with respect to a natural non-integrable almost complex structure on the flag manifold. Further, by considering the relationship between this almost complex structure and the various homogeneous fibrations of the flag manifold over Grassmannians, we are able to generate several harmonic 'Gauss maps' associated to a given harmonic map into a Grassmannian.

These constructions are a special case of a general construction of adapted twistor bundles over Riemannian symmetric spaces, considered in the Appendix to Chapter 6, which is, in turn, a special case of a very general theory of twistor bundles due to Rawnsley [59]. Salamon has considered similar constructions in the case of bundles of almost

CHAPTER 1

MANIFOLDS OF MAPS

In 1956, J. Eells showed that the space $C^0(S,M)$ of continuous maps from a compact Hausdorff space S to a smooth manifold M admits a smooth (in general infinite dimensional) differentiable structure [19]. Since then, many other interesting spaces of maps have been given such structures and a formalism developed to treat the problem (see [30] and [57]).

In this chapter we extend this formalism to include spaces of bounded, but possibly discontinuous maps, in which the smooth maps are not dense and apply it to exhibit differentiable structures (and complete Finsler structures) on some spaces of maps that have arisen in non-linear analysis, in particular in the study of harmonic maps.

A. *Generalised Manifold Models*

Let M be a measure space. Denote by $L^\infty(M,\mathbb{R}^n)$ the totality of essentially bounded measurable functions $M \to \mathbb{R}^n$. $L^\infty(M,\mathbb{R}^n)$ is a Banachable space with a norm given by $\|f\|_\infty = \operatorname*{essup}_{x\in M} f(x)$, $f \in L^\infty(M,\mathbb{R}^n)$.

Definition A is called a *generalised manifold model for M* if, for each n, there is a Banachable space of functions $M \to \mathbb{R}^n$, defined up to sets of measure zero, denoted $A(M,\mathbb{R}^n)$ s.t.

(i) $A(M,\mathbb{R}^n) \subset L^\infty(M,\mathbb{R}^n)$ and the inclusion is continuous.

(ii) $A(M,L(\mathbb{R}^p,\mathbb{R}^q)) \hookrightarrow L(A(M,\mathbb{R}^p), A(M,\mathbb{R}^q))$ and the inclusion is continuous i.e.

$$\|A.\xi\| \le \text{const}\|A\|\ \|\xi\| \quad \text{for} \quad A \in A(M,L(\mathbb{R}^p,\mathbb{R}^q))$$
$$\xi \in A(M,\mathbb{R}^p)$$

(iii) (composition property) Let $\psi : \mathcal{O} \subset \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^s$ be a smooth map from some open set $\mathcal{O} \subset \mathbb{R}^p \times \mathbb{R}^q$ and let $g \in A(M,\mathbb{R}^p)$. Suppose there exists an open set $W \subset A(M,\mathbb{R}^q)$ such that for each $f \in W$ there is a null-set $N_f \subset M$ with $\overline{g \times f(M-N_f)} \subset \mathcal{O}$. Then $\psi(g,f) \in A(M,\mathbb{R}^s)$ and the map $A^g(\psi) : W \to A(M,\mathbb{R}^s)$ given by

$$(A^g(\psi)f)(s) = \psi(g(s),f(s)) \quad \text{a.e} \quad s \in M\,,$$

is continuous.

In fact, such a $A^g(\psi)$ is smooth:

Lemma 1.1 Let A be a generalised manifold model, then the map $A^g(\psi)$ defined above is smooth (C^∞) and $D(A^g\psi) = A^g(d_2\psi)$ where $d_2\psi : \mathcal{O} \to L(\mathbb{R}^q,\mathbb{R}^s)$ is the partial derivative of ψ with respect to $\mathbb{R}^q$.

Proof This now familiar argument is here based on that of Eliasson [30]. It suffices to show that $D(A^g(\psi)) = A^g(d_2\psi)$, then the composition property shows that $A^g(d_2\psi)$ is continuous and hence $A^g(\psi)$ is C^1 and by iterating the argument on successive derivatives of ψ we have that $A^g(\psi)$ is C^∞.

So let $u \in W$ then there is a null set $N \subset M$ with $\overline{g \times u\ (M\setminus N)} \subset \mathcal{O}$ and thus there exists $\varepsilon > 0$ such that if $h \in A(M,\mathbb{R}^q)$ and $\|h\|_A < \varepsilon$, we have

$$(g(s),\ u(s) + h(s)) \in \mathcal{O} \quad \text{a.e.}\ s \in M.$$

For such an h we have

$$\psi(g(s),u(s)+h(s)) - \psi(g(s),u(s)) - d_2\psi(g(s),u(s))h(s) = \theta(g(s),u(s),h(s))h(s) \qquad \text{a.e. } s \in M.$$

with θ given by $\theta(x,u,v) = \int_0^1 d_2\psi(x,u+tv) - d_2\psi(x,u)\,dt$ for $x \in \mathbb{R}^p$ and $u,v \in \mathbb{R}^q$.

Now θ is defined on some open $\mathcal{O}' \subset \mathbb{R}^p \times (\mathbb{R}^q \times \mathbb{R}^q) \to L(\mathbb{R}^q, \mathbb{R}^s)$ with $\overline{g \times (u\times 0)(M\setminus N)} \subset \mathcal{O}'$ thus there is a neighbourhood W' of $u \times 0$ in $L^\infty(M, \mathbb{R}^q \times \mathbb{R}^q)$ and thus in $A(M, \mathbb{R}^q \times \mathbb{R}^q)$ such that if $w \times v \in W'$ there is a null set $N_{w\times v} \subset M$ with $\overline{g \times (w\times v)(M\setminus N_{w\times v})} \subset \mathcal{O}'$. So $A^g(\theta)$ is defined and continuous on W' by the composition property.

Now suppose that $\|h\|_A$ is so small that $(u,h) \in W'$, then

$$\|A^g(\psi)(u+h) - A^g(\psi)(u) - A^g(d_2\psi)(u).h\| = \|A^g(\theta)(u,h).h\| .$$

Now by axiom (ii) $A^g(d_2\psi)(u) \in L(A(M,\mathbb{R}^q), A(M,\mathbb{R}^s))$ and

$$\| A^g(\theta)(u,y).h\| \le \text{const} \| A^g(\theta)(u,h)\| \; \|h\|.$$

$A^g(\theta)(u,0) = 0$, so that by the continuity of $A^g(\theta)$ at $(u,0)$ we have

$D(A^g(\psi))(u) = A^g(d_2\psi)(u)$ and the proof is complete. □

Now let N be a smooth paracompact finite dimensional Riemannian manifold which we assume to be isometrically and properly embedded as a closed submanifold of some $\mathbb{R}^n$ by the theorem of Gromov-Rohlin [37]. Then, if A is a generalised manifold model for M we denote by $A(M,N)$ the set $\{f \in A(M,\mathbb{R}^n) : f(s) \in N \text{ a.e. } s \in M\}$ endowed with the induced topology. Since $A(M,\mathbb{R}^n) \subset L^\infty(M,\mathbb{R}^n)$ and N is closed, $A(M,N)$ is closed.

Theorem 1.2 If A is a generalised manifold model for M, $A(M,N)$ is a closed smooth split submanifold of $A(M,\mathbb{R}^n)$.

Proof We follow Palais [55].

Endow $\mathbb{R}^n$ with a smooth Riemannian metric so that N is isometrically embedded as a totally geodesic submanifold. This can always be done (see eg. Hamilton, p.108 [38]).

Let $\exp : \mathcal{O} \subset T\,\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be the exponential map associated with this metric. Thus exp is a smooth map defined on some neighbourhood $\mathcal{O}$ of the zero section $\mathbb{R}^n \times \{0\}$ in $T\,\mathbb{R}^N$. We also have the following properties: (a) for $x \in \mathbb{R}^n$, $d_2\exp(x,0)$ is an isomorphism,

(b) shrinking $\mathcal{O}$ if necessary, for $x \in N$, $v \in T_xN \cap \mathcal{O}$ iff $\exp v \in N$.

Now let $g \in A(M,N)$ then for some null set $N_g \subset M$, $\overline{g(M\backslash N_g)} \times \{0\} \subset \mathcal{O}$ and is compact, thus there is a neighbourhood W of zero in $A(M,\mathbb{R}^n)$ such that for $u \in W$ there is a null set $N_u \subset M$ with $\overline{g \times u(M\backslash N_u)} \subset \mathcal{O}$ so that we may apply the composition property and Lemma 1.1 to exp to get a smooth map $A^g(\exp) : W \to A(M,\mathbb{R}^n)$. Further, from property (a) of exp we see that $D(A^g(\exp))(0)$ is an isomorphism and so by the inverse function theorem we have a neighbourhood of zero $W' \subset W$ on which $A^g(\exp)$ is a diffeomorphism.

We claim that $(W', A^g(\exp))$ is a split submanifold chart for $A(M,N)$ at g. To do this we must find a suitable splitting of $A(M,\mathbb{R}^n)$. Now $N \times \mathbb{R}^n = TN \oplus V(N)$, the Whitney sum of the tangent and normal bundles on N. Let $\pi, \pi^{\perp} : N \times \mathbb{R}^n \to N \times \mathbb{R}^n$ denote projection onto TN and V(N) respectively and $\tilde{\pi}, \tilde{\pi}^{\perp}$ these projections composed with projections onto the second factor. Then we can extend $\tilde{\pi}, \tilde{\pi}^{\perp}$ to smooth maps $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$.

Then by the composition property $A^g(\tilde{\pi}),\ A^g(\tilde{\pi}^{-1}) : A(M,\mathbb{R}^n) \to A(M,\mathbb{R}^n)$ are bounded linear maps with images denoted $A(g^{-1}TN)$, $A(g^{-1}V(N))$. It is easy to see that $A(M,\mathbb{R}^n)$ splits as a topological direct sum $A(g^{-1}TN) \oplus A(g^{-1}V(N))$ and to see that $A(g^{-1}TN) = \left\{v \in A(M,\mathbb{R}^n) : v(s) \in T_{g(s)}N\right\}$: the space of variations of g.

Further, for $v \in W'$, $A^g(\exp)v \in A(M,N)$ iff $v \in A(g^{-1}TN)$ and so $(W', A^g(\exp))$ is indeed a submanifold chart and the proof is complete. □

Recall the following result of Palais [56].

Theorem 1.3 Let M be a closed submanifold of a Banach space E. Then M has a complete Finsler structure induced by the flat Finsler structure on E.

Corollary $A(M,N)$ admits a complete Finsler structure.

Theorem 1.4 Let N,P be closed embedded manifolds on $\mathbb{R}^p$, $\mathbb{R}^q$ respectively and $\theta : N \to P$ a smooth map. Then the map

$$A(\theta) : A(M,N) \to A(M,P) \text{ given by } A(\theta)(f) = \theta \circ f \text{ is smooth.}$$

Proof Extend θ to a map $\mathbb{R}^p \to \mathbb{R}^q$ and use the composition property and Lemma 1.1 to get a smooth map $A(\theta) : A(M,\mathbb{R}^p) \to A(M,\mathbb{R}^q)$. The result now follows by restricting $A(\theta)$ to $A(M,N)$. □

Corollary If N is embedded in $\mathbb{R}^p$ and $\mathbb{R}^q$, the corresponding manifolds $A(M,N) \subset A(M,\mathbb{R}^p)$, $A(M,N) \subset \mathbb{R}^q$ are diffeomorphic.

We now exhibit some generalised manifold models.
Let M be a finite-dimensional Riemannian manifold.

Denote by $A^{p,1}(M,\mathbb{R}^n)$ the space of essentially bounded functions $M \to \mathbb{R}^n$ with distributional 1st derivatives in L^p and equip it with the norm given by

$$\|f\|^{p,1}_{\infty} = \|f\|_{\infty} + \left(\int_M |df|^p * 1\right)^{1/p}.$$

It is easy to see that $A^{p,1}(M,\mathbb{R}^n)$ is a Banach space. $C^o \cap A^{p,1}(M,\mathbb{R}^n)$, the continuous bounded functions with L^p 1st derivatives, is clearly a closed subspace of $A^{p,1}(M,\mathbb{R}^n)$. It is clear that if M has finite volume, $A^{p,1}(M,\mathbb{R}^n)$ is just $L^p_1 \cap L^\infty(M,\mathbb{R}^n)$ and if M is compact, $C^o \cap A^{p,1}$ is just $C^o \cap L^p_1$.

Theorem 1.5 $L^\infty, A^{p,1}$ and $C^o \cap A^{p,1}$ are all generalised manifold models for M.

Proof *(a)* L^∞: Axiom (ii) is trivial, so let $\psi : \mathcal{O} \subset \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^s$ be a smooth map and let $g \in L^\infty(M,\mathbb{R}^p)$, $W \subset L^\infty(M,\mathbb{R}^q)$ be as in the hypotheses of the composition property. Then if $u \in W$, there is a null-set $N_u \subset M$ s.t. $\overline{g \times u(M\backslash N_u)} \subset \mathcal{O}$ and is compact. Thus there is an $\varepsilon > 0$ and a compact $K \subset \mathcal{O}$ s.t. if $h \in W$ and $\|u-h\| < \epsilon$ then $g \times h(s) \in K$ a.e. $s \in M$. Now ψ is uniformly continuous on K and so the continuity of $L^\infty_g(\psi)$ follows.

(b) $A^{p,1}$: Axiom (ii) is straightforward using the Leibniz rule. For axiom (iii), we have already shown that composition is continuous in L^∞ so it only remains to check the behaviour of the derivatives:

$d\psi(g \times w) = d_1\psi(g \times w).dg + d_2\psi(g \times w).dw$ so if $w_n \to w$ in $A^{p,1}$ then by the above:

$d_i\psi(g,w_n) \to d_i\psi(g,w)$ in L^∞ $i = 1,2$ and the result follows since multiplication $L^\infty \times L^p \to L^p$ is continuous.

(c) $C^0 \cap A^{p,1}$: it suffices to observe that composition by a smooth function preserves C^0 and hence, by the above, $C^0 \cap A^{p,1}$. □

Remarks *(i)* That $L^\infty(M,N)$ is a smooth manifold is a result of Krikorian [47] and it is easy to see that L^∞ is a generalised manifold model for any measure space.

(ii) It is easy to see that $C^0 \cap A^{p,1}(M,N)$ is a closed submanifold of $A^{p,1}(M,N)$. (Here we do not require that the submanifold model space split the ambient model space.)

(iii) If M is compact and $p > \dim M$, our results on $A^{p,1}$, $C^0 \cap A^{p,1}$ reduce to the well-known fact that L_1^p is a manifold model in the sense of Eliasson by the Sobolev theorem. However, the induced Finsler structures, and thus the induced "differential geometry", are quite different.

(iv) The smoothness requirements on M and N can be lowered considerably. It is clear that M need only be sufficiently differentiable that the generalised manifold models be well-defined, while a loss of smoothness in N is reflected by a corresponding loss of smoothness in $A(M,N)$ since the differentiability of the charts is a consequence of the differentiability of the exponential map on N (see Eliasson for precise statements [30]).

(v) If $A = L^\infty$, or if the smooth maps are dense in A, we can provide differentiable structures for $A(M,N)$ by the intrinsic method of Eliasson [30], without recourse to embedding N in some $\mathbb{R}^n$. However, in the case of $A^{p,1}$ $(p < \dim M)$, it is unclear how to define the tangent spaces of $A^{p,1}(M,N)$ at discontinuous maps without recourse to an embedding.

(vi) With the aid of the Palais multiplication lemma [57] it can be shown that the following spaces give rise to generalised manifold models for M compact:

(a) $B^{\infty,p}_k$: the space of functions with essentially bounded derivatives up to order k-1 with kth derivatives in L^p with norm
$$\sum_1^{k-1} \|d^i f\|_\infty + \|d^k f\|_{L^p}.$$

(b) $L^\infty \cap L^{n/k}_k(M,\mathbb{R}^m)$ where $n = \dim M$ and the norm is $\|\cdot\|_\infty + \|\cdot\|_{L^{n/k}_k}$; and also the corresponding spaces of continuous (continuously differentiable) functions.

B. $L^\infty \cap L^2_1(M,N)$ *and harmonic maps*

Henceforth (M,g) will denote a compact Riemannian manifold. In this case the generalised manifold models $A^{2,1} = L^\infty \cap L^2_1$ and $C^0 \cap A^{2,1} = C^0 \cap L^2_1$ are of particular interest in the theory of harmonic maps; the regularity theory of Schoen-Uhlenbeck and Giaquinta-Giusti [64,32,33] shows that every energy-minimising harmonic map in $L^\infty \cap L^2_1(M,N)$ is smooth under certain geometric conditions on the range manifold (see chapter 2), while Hildebrandt *et al.* [42] have shown that every harmonic map in $C^0 \cap L^2_1(M,N)$ is smooth (cf Theorem 1.8 below). Thus these manifolds appear to be the appropriate spaces to do some sort of critical point theory for the energy functional and so we now examine some aspects of the theory of harmonic maps from this point of view.

As usual, we will assume that N is properly and isometrically embedded in some $\mathbb{R}^n$.

Recall that the tangent space to $L^\infty \cap L_1^2(M,N)$ at ϕ is given by

$$\left\{ v \in L^\infty \cap L_1^2(M,\mathbb{R}^n) \; : \; v(x) \in T_{\phi(x)}N \quad \text{a.e.} \quad x \in M \right\}$$

and the Finsler norm of a tangent vector v at ϕ is given by

$$\|v\|_\phi = \left(\int_M g^{ij} \frac{\partial v^a}{\partial x_i} \frac{\partial v^a}{\partial x_j} * 1_M \right)^{\frac{1}{2}} + \operatorname*{essup}_{x \in M} |v(x)|$$

$L^\infty \cap L_1^2(M,N)$ also inherits a Riemannian structure (and hence a Levi-Civita connection) from $L^\infty \cap L_1^2(M,\mathbb{R}^n)$ given by

$$<v,w>_\phi = \int_M \sum_a v^a w^a * 1_M \; .$$

We summarise the situation in the following

Proposition 1.6 $L^\infty \cap L_1^2(M,N)$ is a smooth, infinite dimensional manifold with natural Finsler and L^2 Riemannian structures, containing $C^0 \cap L_1^2(M,N)$ as a closed, separable submanifold. Further:

(i) $C^\infty(M,N)$ is dense in $C^0 \cap L_1^2(M,N)$

(ii) the components of $C^0 \cap L_1^2(M,N)$ are precisely the homotopy classes of $C^0 \cap L_1^2(M,N)$

(iii) if $\dim M \geq 2$, $L^\infty \cap L_1^2(M,N)$ is not separable.

Proof Assertion (i) is proved by a standard mollification argument and implies the separability of $C^0 \cap L_1^2(M,N)$.

Assertion (ii) is a trivial consequence of the continuity of the inclusions $C^\infty(M,N) \to C^0 \cap L_1^2(M,N) \to C^0(M,N)$.

For (iii), first assume $\dim M = m \geq 3$.

Let $\{B_n\}$ be a sequence of disjoint open balls in M where the radius of B_n is $\frac{1}{2^n}$ and for each n let $\phi_n : M \to \mathbb{R}$ be a smooth function with support contained B_n, $0 \leq \phi_n \leq 1$, $\phi_n \equiv 1$ on some open $U_n \subset B_n$ and $\sup_{x \in B_n} |\nabla\phi_n(x)| \leq K2^n$ where K is a constant independent of n. Now let $\{\phi_{n_k}\}$ be a subsequence of the $\{\phi_n\}$ and let $s_r = \sum_{k=1}^{r} \phi_{n_k}$.

Then

$$\|s_r\|_\infty = 1 \quad \text{and} \quad \int_M |ds_r|^2 \leq \sum_{k=1}^{r} \int_{B_{n_k}} |\nabla\phi_{n_k}|^2 \leq C \sum_{n=1}^{n_r} \frac{2^{2n}}{2^{mn}} \leq C \sum_{n=1}^{n_r} \frac{1}{2^{(m-2)n}} \leq C.$$

Thus $\{s_r\}$ is uniformly bounded in L_1^2 (M is compact!) and so a subsequence $\{s_r\}$ converges to some s weakly in L_1^2 and pointwise almost everywhere, thus $s(x) = 1$ a.e. $x \in U_{n_k}$ and $s(x) = 0$ a.e. $x \notin \bigcup_k B_{n_k}$.

It is clear that we may sum any subsequence $\{\phi_{n_k}\}$ in this manner and if s_1, s_2 are the sums corresponding to different subsequences, s_1, s_2 disagree on some B_n whence $\|(s_1 - s_2)|U_n\|_\infty = 1$ and so $\|s_1 - s_2\|_{L^\infty \cap L_1^2} \geq 1$.

Thus we have generated a discrete uncountable subset of $L^\infty \cap L_1^2\,(M, \mathbb{R})$ which is therefore not separable. It easily follows that $L^\infty \cap L_1^2(M,N)$ is not separable.

If $\dim M = 2$, the proof is rather more involved and is therefore relegated to an appendix. $\square$

Now for $\phi \in L^\infty \cap L_1^2(M,N)$ recall that the *energy* of ϕ is given by

$$E(\phi) = \frac{1}{2} \int_M g^{ij} \frac{\partial \phi^a}{\partial x_i} \frac{\partial \phi^a}{\partial x_j} * 1_M.$$

So $E : L^\infty \cap L_1^2(M,N) \to \mathbb{R}$ is the restriction of a continuous quadratic form on $L^\infty \cap L_1^2(M,\mathbb{R}^n)$ and is therefore smooth.

Let $v \in T_\phi(L^\infty \cap L_1^2(M,N)) = L^\infty \cap L_1^2(\phi^{-1}TN)$, then

$$dE_\phi(v) = \int_M \langle d\phi, dv \rangle * 1$$

$$= \int_M \langle d\phi, \nabla v \rangle * 1$$

where ∇v is the component of dv in $\phi^{-1}TN$. ∇v can be identified with the co-variant derivative of v in $\phi^{-1}TN$ if ϕ is sufficiently smooth. Thus we conclude that the critical points of E are precisely the weakly harmonic maps in $L^\infty \cap L_1^2(M,N)$ which we denote by $\mathrm{Harm}(M,N)$. It is immediate from the smoothness of E that $\mathrm{Harm}(M,N)$ is closed. By results of Hildebrandt [42] we have

$$\mathrm{Harm}(M,N) \cap C^0(M,N) = \mathrm{Harm}(M,N) \cap C^\infty(M,N).$$

However, as many authors have remarked

$$\phi : B^n \to S^{n-1} \quad \text{given by} \quad \phi(x) = \frac{x}{|x|} \text{ is in } \mathrm{Harm}(B^n, S^{n-1})$$

so that $\mathrm{Harm}(M,N) \cap C^0(M,N) \neq \mathrm{Harm}(M,N)$.

Proposition 1.7 $\mathrm{Harm}(M,N) \cap C^0(M,N)$ is locally compact in $L^\infty \cap L_1^2(M,N)$ and open in $\mathrm{Harm}(M,N)$.

The proof is based on the following *a priori* estimate of Hildebrandt-

Giaquinta [34]:

Theorem 1.8 Let U be a compact manifold with boundary and $\phi \in \mathrm{Harm}(U,N)$ with $\phi(U)$ contained in a regular ball $B_R(y)$ in N. Then $\phi \in \mathrm{Harm}(U,N) \cap C^0(U,N)$ and for each open $\Omega \subset\subset U$ there is a number c depending only on Ω, dim U, R and the C^2 norms of the metrics on U and $B_R(y)$ s.t.

$$\sup_{x\in\Omega} |d\phi(x)| \leq c.$$

Here a regular ball $B_R(y)$ is a geodesic ball disjoint from the cut-locus of y with $R < \frac{\pi}{2\sqrt{k}}$ where $k > 0$ is an upper bound for the sectional curvatures on $B_R(y)$.

Proof of Proposition 1.7 Let $\phi \in \mathrm{Harm}(M,N) \cap C^0(M,N)$ and cover $\phi(M)$ with open balls $B_{R_y}(y)$, $y \in \phi(M)$ so that each $B_{2R_y}(y)$ is regular. Since M is compact, so is $\phi(M)$ so we take $y_1,\ldots y_n \in \phi(M)$ such that $B_{R_1}(y_1),\ldots,B_{R_n}(y_n)$ cover $\phi(M)$ and $B_{2R_i}(y_i)$ is regular for each i. Let $U_i = \phi^{-1}(B_{R_i}(y_i))$, then the U_i form a finite open cover of M and there are $x_1\ldots x_m \in M$, $\delta_1\ldots\delta_m$ s.t. $\bigcup_i^m B_{\delta_j}(x_j) = M$ and each $B_{\delta_j}(x_j) \subset\subset U_i$ some i.

Now let $R = \min_{1\leq i\leq n} R_i$ and let $\mathcal{U} = \{u \in L^\infty \cap L_1^2(M,N) : \|u-\phi\|_\infty < R\}$.

Then $\mathcal{U}$ is open in $L^\infty \cap L_1^2(M,N)$, and if $u \in \mathcal{U} \cap \mathrm{Harm}(M,N)$, $u(U_i) \subset B_{2R_i}(y_i)$ and so is continuous on U_i by theorem 1.8. Thus $\mathcal{U} \cap \mathrm{Harm}(M,N) \subset C^0(M,N) \cap \mathrm{Harm}(M,N)$ and so the latter is open in $\mathrm{Harm}(M,N)$.

Further there is a number c_j s.t. $\|du\|_\infty < c_j$ on $B_{\delta_j}(x_j)$ and thus $\mathcal{U} \cap \mathrm{Harm}(M,N)$ is bounded in $C^1(M,N)$.

We have the classical estimate of Agmon [2]:

$$\|u\|_{L_2^{2p}} \le c(p)\left(\|\Delta u\|_{L^{2p}} + \|u\|_{L_1^{2p}}\right)$$

where Δ is the Laplace-Beltrami operator on M. Since u is harmonic, the C^0 norm of Δu is bounded in terms of the C^1 norm of u and thus $U \cap \text{Harm}(M,N)$ is bounded in L_2^{2p} for all p. Taking p large enough and applying the Sobolev-Rellich-Kondrachov embedding theorems shows that $U \cap \text{Harm}(M,N)$ is compact in $C^1(M,N)$ and hence in $L^\infty \cap L_1^2(M,N)$. □

Now let ϕ be a continuous (and hence smooth) critical point of E and consider the Hessian:

$$d^2E_\phi \; : \; L^\infty \cap L_1^2(\phi^{-1}TN) \times L^\infty \cap L_1^2(\phi^{-1}TN) \to \mathbb{R}.$$

Since ϕ is a critical point, this can be identified with the second fundamental form of E w.r.t. the L^2 Levi-Civita connection on $L^\infty \cap L_1^2(M,N)$.

A calculation in [23] shows that

$d^2E_\phi(v,w) = \int_M \langle J^\phi v,w\rangle$ where J^ϕ is a second order positive linear elliptic operator on $\phi^{-1}TN$, the *Jacobi operator*, given by

$J^\phi = \Delta^\phi - \text{Trace}\; R^N(d\phi,.)d\phi$. Here $\Delta\phi$ is the Laplacian on sections of $\phi^{-1}TN$.

Writing $d^2E_\phi(v,w)$ as

$$\int_M \langle\nabla u,\nabla w\rangle - \langle\text{Trace}\; R^N(d\phi,v)d\phi,w\rangle \quad * \; 1_M$$

it is clear that d^2E_ϕ extends to a continuous quadratic form

$$L_1^2(\phi^{-1}TN) \times L_1^2(\phi^{-1}TN) \to \mathbb{R}.$$

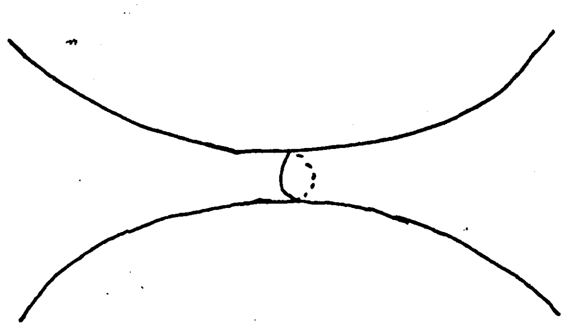
Say that ϕ is *non-degenerate* if this extended quadratic form is non-degenerate.

The ellipticity and positivity of J^ϕ ensure that the spectrum of J^ϕ is discrete and bounded below and that the eigenspaces of J^ϕ are finite-dimensional and smooth.

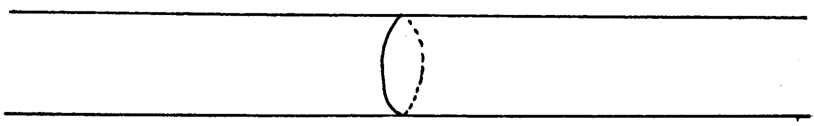
The *index* of ϕ is the sum of the multiplicities of the negative eigenvalues and the *nullity* of ϕ is the dimension of $\operatorname{Ker} J_\phi$. If ϕ has index zero, it is said to be *stable*. It is clear that ϕ is non-degenerate iff ϕ has nullity zero.

Remarks

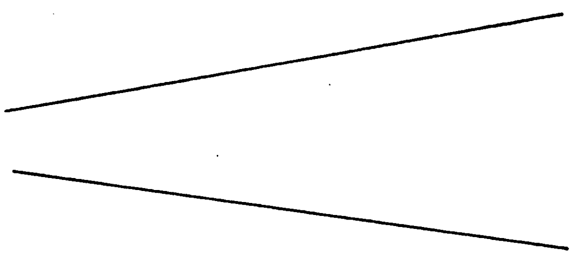
(i) If the metrics on M and N are changed, the differentiable structure of $L^\infty \cap L_1^2(M,N)$ remains fixed (theorem 1.4), but the Finsler geometry changes and so does the energy function. The critical point theory of the energy; number of critical points, their index and nullity, stability and non-degeneracy and so on, is sensitive to changes in the metric as the following example of Sampson demonstrates for harmonic maps of S^1 (geodesics) into a cylinder whose metric varies:



1. *A stable critical point which is non-degenerate up to rotations of S^1.*



2. *A stable but degenerate critical point.*



3. *No critical points.*

For results concerning the behaviour of harmonic maps as the metrics are deformed see [21].

(ii) Another topic of interest is the question of removable singularities for harmonic maps: if $\phi \in \mathrm{Harm}(M\backslash S,N)$, under what conditions on S is $\phi \in \mathrm{Harm}(M,N)$, for a characterisation in terms of capacity see [24].

(iii) Lemaire has pointed out that E does not, in general, satisfy Palais-Smale condition C on $L^\infty \cap L_1^2(M,N)$ since there are examples where E does not achieve its infinum on a component: e.g. $\inf_{\phi\in\mathcal{H}} E(\phi) = 0$ for $\mathcal{H}$ the component of the identity map in $L^\infty \cap L_1^2(S^n,S^n)$, $n \geq 3$.

Appendix 1: Non-Separability of $L^\infty \cap L_1^p$ *in the Sobolev limit*

Proposition A.1: Let $\dim M = n$, then $L^\infty \cap L_1^n(M,N)$ is non-separable. ($L^\infty \cap L_1^n$ is equipped with norm $\|u\|_\infty + \left(\int_M |du|^n)^{1/n}\right)$).

The proof below was suggested to me by Professor J.F. Toland.

Proof It suffices to consider $M = B_1^n$, the unit n-ball, and $N = \mathbb{R}$. Let $\phi : B_1^n \to \mathbb{R}$ be an unbounded function with support contained in B_1^n and finite L_1^n norm (eg. $\phi(r,\theta) = \psi(r)\log\log\frac{4}{r}$ where $\psi \equiv 1$ $r < \frac{1}{2}$, $\psi \equiv 0$ $r > \frac{3}{4}$ see Adams [1]).

Let $\phi_r(x) = \phi(\frac{x}{r})$. Then ϕ_r has support contained in B_r^n and

$$\int_{\mathbb{R}^n} |d\phi_r|^n dx = \int_{B_r} \frac{1}{r^n}\ |d\phi(\tfrac{x}{r})|^n dx = \int_{B_1} \frac{r^n}{r^n} |d\phi(x')|\,dx' = \int_{\mathbb{R}^n} |d\phi|^n dx \quad .$$

Now define $\phi^k : B_1^n \to \mathbb{R}$ by

$$\phi^k(x) = \phi(x) \qquad \text{if } \phi(x) \le k$$

$$\phi^k(x) = k \qquad \text{if } \phi(x) \ge k.$$

Then $\phi^k \in L^\infty \cap L_1^n(B_n^1, \mathbb{R})$, $\|\phi^k\|_\infty = k$ and $\|d\phi^k\|_{L^n} \le \|d\phi\|_{L^n} = M$ say.

Thus $\dfrac{\phi_r^k}{k} \in L^\infty \cap L_1^n(B_n^1, \mathbb{R})$ with support in B_r, $\|\dfrac{\phi_r^k}{k}\|_\infty = 1$ and $\|\dfrac{d\phi_r^k}{k}\|_{L^n} \le \dfrac{M}{k}$.

Thus given a disjoint sequence of balls $\{B_n\}$ we can find $\phi_n \in L^\infty \cap L_1^n(B_1, \mathbb{R})$ with $\text{support}(\phi_n) \subset B_n$, $\|d\phi_n\|_{L^n} \le \frac{M}{2^n}$, $\|\phi_n\|_\infty = 1$ and the proof of of non-separability proceeds exactly as in Proposition 1.6. □

Remark Analysis in L_1^n has many interesting properties: Schoen-Uhlenbeck [65] have shown that $C^\infty(M,N)$ is dense in $L_1^2(M,N)$ (thought of as a closed subset of $L_1^2(M,\mathbb{R}^n)$) --- if dim $M \le 2$.* Their methods apply equally to show that $C^\infty(M,N)$ is dense in $L_1^n(M,N)$ for dim $M = n$.

* This is not always true if dim $M \geq 3$.

CHAPTER 2

HARMONIC MAPS OF FINITE ENERGY FROM NON-COMPACT MANIFOLDS

In this chapter, we establish the existence of harmonic maps of finite energy from non-compact Riemannian manifolds into certain Riemannian manifolds. This question has been studied by Schoen-Yau [66] in the case where the range has non-positive sectional curvatures and by Lemaire [49] in the case of two-dimensional domain.

The method of these previous authors required *a priori* bounds on C^1 or Holder norms of harmonic maps to ensure convergence of a suitable sequence of harmonic maps. We have recourse to the powerful regularity theory of Schoen-Uhlenbeck [64], which does not provide *a priori* estimates and so we must use a different technique to establish the existence of a harmonic map. It turns out that the necessary technique is a simple variant of the direct method of the calculus of variations.

Henceforth all manifolds will be assumed to be smooth (C^∞) connected, finite-dimensional and endowed with a fixed Riemannian metric.

1. *Spaces of Maps*

Let M be a manifold, possibly with boundary. Denote by $L_1^2(M,\mathbb{R}^K)$ the space of square integrable functions from M into $\mathbb{R}^K$ with square integrable first derivatives in the sense of distributions. $L_1^2(M,\mathbb{R}^K)$ is a Hilbert space with inner product given by

$$\langle f,g\rangle_{1,2} = \int_M \langle f,g\rangle dv + \int_M \langle df,dg\rangle dv \ ,$$

where dv is the volume element on M and the inner product in the second integral is the Hilbert-Schmidt inner product on $\mathrm{Hom}(TM, \mathbb{R}^K)$.

Denote by $L^2_{1,0}(M,\mathbb{R}^K)$ the closure in $L^2_1(M,\mathbb{R}^K)$ of the space of smooth functions with compact support.

Denote by $L^2_{1,loc}(M,\mathbb{R}^K)$ the space of functions whose restriction to any open set U with compact closure in M are in $L^2_1(U,\mathbb{R}^K)$. It is clear that these three spaces coincide if M is compact and without boundary. Recall that the *energy* of a map $f \in L^2_1(M,\mathbb{R}^K)$, denoted $E(f)$, is given by

$$E(f) = \tfrac{1}{2} \int_M \langle df, df \rangle dv.$$

Lemma 1.1 Let M,N be compact manifolds. If $f \in L^2_1(M \times N, \mathbb{R}^K)$, then $f(x,\cdot) \in L^2_1(N,\mathbb{R}^K)$ for a.e $x \in M$ and conversely.

Proof By Fubini's theorem we have that $f(x,\cdot)$ and $df(x,\cdot)$ are square integrable on N for a.e $x \in M$. We must show that the weak derivatives of $f(x,\cdot)$ coincide with the restrictions of df. It suffices to work locally, so let $(U,x_1,\ldots,x_m)$, $(V,y_1,\ldots,y_n)$ be co-ordinate charts on M and N respectively, and let η,ψ be C^∞ functions with supports contained in U and V respectively.

Then

$$\int_{U\times V} \eta(x)\psi(y)\frac{\partial f}{\partial y_i}\, dxdy = - \int_{U\times V} \eta(x)\frac{\partial \psi}{\partial y_i}(y)\; f(x,y)dxdy.$$

Thus

$$\int_U \eta(x)\left(\int_V \psi(y)\frac{\partial f}{\partial y_i} + \frac{\partial \psi}{\partial y_i}(y)\; f\, dy\right) dx = 0$$

so that, since ψ,η were arbitrary, for a.e $x \in M$, $\frac{\partial f}{\partial y_i}(x,\cdot)$ is the i^{th} weak derivative of $f(x,\cdot)$ and the lemma follows. □

We now collect the necessary convergence and semi-continuity results about $L^2_1(M,\mathbb{R}^K)$. Proofs may be found in Morrey [52].

Lemma 1.2 Let M be a compact manifold and let $\{f_n\}$ be a bounded sequence in $L^2_1(M,\mathbb{R}^K)$. Then $\{f_n\}$ has a subsequence $\{f_j\}$ which converges to some $f \in L^2_1(M,\mathbb{R}^K)$ weakly in L^2_1, strongly in L^2 and pointwise almost everywhere. Further

$$E(f) \leq \liminf_{j\to\infty} E(f_j).$$

Now let N be a complete manifold which we assume to be isometrically and properly embedded in some $\mathbb{R}^K$ by the theorem of Gromov-Rohlin. Let $L^2_1(M,N)$ denote $\left\{f \in L^2_1(M,\mathbb{R}^K) : f(x) \in N \text{ for a.e } x \in M\right\}$ and define $L^2_{1,0}(M,N)$ and $L^2_{1,loc}(M,N)$ similarly. It is clear that $L^2_1(M,N)$ is a closed subset of $L^2_1(M,\mathbb{R}^K)$.

Recall that a C^2 map $f : M \to N$ is said to be *harmonic* if f extremises the energy with respect to all compactly supported variations in $C^\infty(M,N)$.

Definition A Riemannian manifold N is said to be *homogeneously regular* if there exist positive constants C_1, C_2 such that any point of N is in the domain of a co-ordinate chart $\theta : V \to \mathbb{R}^n$ whose image is the unit ball and

$$C_1 |d\theta_y(Y)|^2_{\mathbb{R}^n} \leq |Y|^2_y \leq C_2 |d\theta_y(Y)|^2_{\mathbb{R}^n}$$

for any $y \in V$, $Y \in T_yN$.

In particular, any compact manifold is homogeneously regular.

We record a result of Morrey [51,52] that will be useful in the next section.

Theorem 1.3 Let N be homogeneously regular and let $\phi \in L_1^2(D^2, N)$ with continuous boundary values in the L_1^2 sense. Then there is a harmonic map $f : D^2 \to N$, smooth on the interior, continuous on D^2 such that

$$f|bD^2 = \phi|bD^2 \ a.e.$$

2. *Induced Maps on* $\pi_1(M)$

Following Schoen-Yau [68], we define a map on fundamental groups induced by a map $f \in L_1^2(M,N)$.

Let M^{m+1} be a compact m+1 dimensional manifold and let $\gamma_1, \ldots, \gamma_\ell$ be embedded curves in M^{m+1} which form a generating set for $\pi_1(M^{m+1}, *)$. We assume that each γ_i is defined on $[-2,2]$, $\gamma_i(0) = *$ each i and that $\gamma_i = \gamma_j$ on $(-1,1)$ for $1 \le i, j \le \ell$.

Let T_i be a tubular neighbourhood of γ_i in M^{m+1} such that $\psi_i : S^1 \times I^m \to T_i$ is a smooth immersion. Here I^m is the unit m-cell and S^1 is thought of as $[-2,2] \bmod \{-2,2\}$.

For $s \in I^m$, let $\gamma_i^s : S^1 \to M$ be the curve given by $\gamma_i^s(t) = \psi_i(t,s)$ and assume that γ_i^s and γ_j^s agree on $(-1,1)$ for $1 \le i,j \le \ell$ and that $\gamma_i^0 = \gamma_i$ for each i.

Lemma 2.1 Let $f \in L_1^2(M^{m+1}, N)$. Then f may be taken to be continuous on γ_i^s for a.e $s \in I^m$.

Proof Using ψ_i to identify $L_1^2(T_i, N)$ with $L_1^2(S^1 \times I^m, N)$, the result is immediate from Lemma 1.1 and Sobolev's theorem. □

Lemma 2.1.1 Let $f \in L^2_1(M,N)$. Then f may be represented by a map, also called f, with the following property (the A.C. property);

For each co-ordinate chart on M, $(U, x_1, \ldots, x_m)$ and each $\alpha : 1 \le \alpha \le m$ f is absolutely continuous in x_α for almost all values of the other variables.

Further this representative may be defined almost everywhere as the Lebesgue derivative of the set function

$$\Omega \to \int_\Omega \mathit{f}\, dx \quad ,$$

where f is any representative of f.

Proof Morrey [52], Lemma 3.1.8. □

Henceforth we shall work exclusively with this representative.

Proposition 2.2 [68] Let $\dim M = 2$ and $f \in L^2_1(M,N)$. For $s, t \in I$, if f is continuous on γ_i^s and γ_i^t then $f(\gamma_i^s)$ is homotopic to $f(\gamma_i^t)$.

Proof Identifying T_i with $S^1 \times I$, for almost all $\theta \in S^1$, f is continuous on the ray $\theta = \text{constant}$, which we denote by δ_θ. For such a θ, f is continuous on the path γ given by

$$\gamma = \gamma_i^s \circ \delta_\theta \circ (\gamma_i^t)^{-1} \circ (\delta_\theta)^{-1}$$

which bounds an immersed disc on which f is L^2_1. Further, since f has the A.C. property, the L^2_1 boundary values of f on γ coincide almost everywhere with the restriction of f to γ. Thus by theorem 1.3, $f(\gamma)$ bounds a disc in N and so is contractible. The proposition follows immediately. □

Remark By choosing the same $\theta \in (-1,1)$ for each i, we observe that there is a curve σ from $\gamma_i^s(0)$ to $\gamma_i^t(0)$ on which f is continuous so that the based homotopy classes $[f(\gamma_i^t)]$ and $[f(\gamma_i^s)]$ are conjugate via $f(\sigma)$ for each i.

The following generalisation has been claimed by Schoen [63] but there seems to be no proof of it in the literature.

Proposition 2.3 Let $f \in L_1^2(M^{m+1},N)$. Then there is a co-null subset $I_f^m \subset I^m$ such that for $s \in I_f^m$, f is continuous on γ_i^s and if $s,t \in I_f^m$, then $f(\gamma_i^s)$ is homotopic to $f(\gamma_i^t)$ for each i.

Proof As before we identify T^i with $S^1 \times I^m$.

By lemma 1.1 there exist null-sets $N^j \subset I^j$, $1 \le j \le m$ such that for $x \in I^j \backslash N^j$,

$$f(\cdot,x) \in L_1^2(S^1 \times I^{m-j}) .$$

In particular, for $x \in I^m \setminus N^m$, $f(\cdot,x)$ is continuous on S^1. Further, since f has the A.C. property on $S^1 \times I^m$, it is easy to see that there is a null set $\overline{N}^j \supset N^j$ such that for $x \in I^j \setminus \overline{N}^j$

$$f(\cdot,x) \text{ has the A.C. property on } S^1 \times I^{m-j} .$$

Define I_f^m by

$I_f^m = \{(x_1,\dots,x_m) \in I^m : f(\cdot,x_{j+1},\dots,x_m) \in L_1^2(S^1 \times I^j)$ and has the A.C. property on $S^1 \times I^j$ for each $j : 1 \le j \le m\}$.

Since $I_f^m \supset \bigcap_j I^{m-j} \times (I^j \setminus \overline{N}^j)$, it is clear that I_f^m is co-null.

We prove the proposition by induction on m, the case $m=1$ being true by proposition 2.2.

Suppose now the proposition is true for m-1.

Let $x = (x_1,\dots,x_m)$, $y = (y_1,\dots,y_m) \in I_f^m$. Then $f(\cdot, x_m)$, $f(\cdot, y_m)$ are L_1^2 on $S^1 \times I^{m-1}$ and have the A.C. property there. Further, we have $x' = (x_1,\dots,x_{m-1}) \in I^{m-1}_{f(\cdot,x_m)}$, $y' = (y_1,\dots,y_{m-1}) \in I^{m-1}_{f(\cdot,y_m)}$ so by the induction hypothesis, for a.e. $z \in I^{m-1}$, $f(\cdot, z, x_m)$ and $f(\cdot, z, y_m)$ are continuous on S^1 and we have

$$f(\cdot, z, x_m) \simeq f(\cdot, x), \quad f(\cdot, z, y_m) \simeq f(\cdot, y)$$

Now, arguing as above, for a.e. $z \in I^{m-1}$, $f(\cdot, z, \cdot)$ is L_1^2 and has the A.C. property on $S^1 \times I$ so that

$$f(\cdot, z, x_m) \simeq f(\cdot, z, y_m) \text{ for a.e. } z \in I^{m-1},$$

by Proposition 2.2. Thus $f(\cdot, z) \simeq f(\cdot, y)$ and the proposition follows. □

Again with a little more work, we can see that for $s,t \in I_f^m$ there is a curve σ from $\gamma_i^s(0)$ to $\gamma_i^t(0)$ on which f is continuous so that the based homotopy classes $[f(\gamma_i^s)]$ and $[f(\gamma_i^t)]$ are conjugate by $f(\sigma)$ for all i.

Now let α be a contractible curve in M^{m+1}. Then α is the boundary of an immersed 2-disc in M^{m+1}. Take a tubular neighbourhood T about this disc so that $\psi : D^2 \times I^{m-1} \to T$ is an immersion and define the path $\gamma^{r,x}$ by

$$\gamma^{r,x}(\theta) = \psi(r,\theta,x)$$

where (r,θ) are polar co-ordinates on D^2 and $x \in I^{m-1}$.

Proposition 2.4 If $f \in L_1^2(M^{m+1}, N)$, then f is continuous on $\gamma^{r,x}$ and $f(\gamma^{r,x})$ is contractible for a.e. $(r,x) \in [\frac{1}{2},1] \times I^{m-1}$.

Proof Identify T with $D^2 \times I^{m-1}$.

As in proposition 2.3 we have a co-null set. $I_f \subset [\frac{1}{2},1] \times I^{m-1}$ such that for (r,x), $(s,y) \in I_f$, $f(\cdot,x)$ is L_1^2 and has the A.C. property on D^2 and $f(r,\cdot,x)$, $f(s,\cdot,y)$ are continuous and mutually homotopic. An application of theorem 1.3 now shows that $f(r,\cdot,x)$ is contractible for all $(r,x) \in I_f$ and the proposition is proved. □

We are now in a position to define the map f induced on the fundamental group of M by $f \in L_1^2(M,N)$:

Fix $s_o \in I_f^m$ and set $*_f = \gamma_i^{s_o}(0)$.

Define $f_\# : \pi_1(M,*_f) \to \pi_1(N,f(*_f))$ by

$$f_\#[\gamma_i^{s_o}] = [f(\gamma_i^{s_o})] \quad \text{for } 1 \le i \le \ell$$

on the generators and extend $f_\#$ so that it is a group homomorphism. Proposition 2.4 ensures that this homomorphism is well-defined. Further, if we choose another $s \in I_f^m$, then by proposition 2.3 and the remarks following it, the corresponding homormorphisms are conjugate.

Of course, if f is continuous, $f_\#$ is the usual map on $\pi_1(M)$.

3. *Existence of a weak solution*

Let M be a complete non-compact manifold and let N be a compact manifold. Exhaust M by an increasing sequence of compact manifolds with boundary; $\{M_n\}_n$.

Let $\phi \in C^\infty(M,N)$ have finite energy and define $\mathcal{H}_\phi$ as follows:

$$\mathcal{H}_\phi = \left\{ f \in L^2_{1,\mathrm{loc}}(M,N) : \text{for each } n,\ (f|M_n)_\# = \tau_n^{-1}(\phi|M_n)_\#\tau_n \right.$$
$$\left. \text{for some path } \tau_n \text{ from } f(*_f) \text{ to } \phi(*_f) \right\} .$$

It is clear that $\mathcal{H}_\phi$ contains all C^∞ maps from M to N that are homotopic to ϕ on compacta. Since $\phi \in \mathcal{H}_\phi$, $\mathcal{H}_\phi$ is non-empty and contains maps of finite energy. Let I be given by

$$I = \inf\{E(f) : f \in \mathcal{H}_\phi\} .$$

Proposition 3.1 There is a map $f_o \in \mathcal{H}_\phi$ such that $E(f_o) = I$.

Proof Let $\{f_i\}$ be a minimising sequence for the energy in $\mathcal{H}_\phi$. Consider $\{f_i|M_n\}_i$ for some n. Since N is compact and $\{E(f_i)\}$ is bounded, it is clear that $\{f_i|M_n\}_i$ is uniformly bounded in $L^2_1(M_n,\mathbb{R}^K)$. So, by Lemma 1.2, a subsequence $\{f_j|M_n\}_j$ converges to some $f^{(n)} \in L^2_1(M_n,\mathbb{R}^K)$ weakly in L^2_1, strongly in L^2 and pointwise almost everywhere. The pointwise convergence ensures that $f^{(n)} \in L^2_1(M_n,N)$ and we have

$$\bar{E}(f^{(n)}) \le \liminf_{j\to\infty} E(f_j|M_n) .$$

Now if $\gamma_1,\ldots,\gamma_\ell$ are generating curves for $\pi_1(M_n,*)$ with tubular neighbourhoods $T_1,\ldots,T_\ell$ which we identify with $S^1 \times I^m$, then the uniform boundedness of $\{E(f_j)\}$ together with Fatou's Lemma shows that for a.e. $s \in I^m$, there exists a number K_s such that

$$\int_{S^1} |df_j(t,s)|^2 dt \le K_s$$

for infinitely many j.

Thus, from the compactness of N, Rellich's theorem and Sobolev's theorem, we get a subsequence $\{f_j\}$ converging uniformly on γ_i^s, for each i. So, for a.e. $s \in I^m$, there is a j such that $f_j(\gamma_i^s)$ is uniformly close to and hence homotopic to $f^{(n)}(\gamma_i^s)$. Thus

$$f^{(n)}{}_{\#} = \tau_n^{-1}(\phi|M_n)_{\#}\tau_n \quad \text{for some } \tau_n. \tag{1}$$

Now using a diagonal argument, we choose a subsequence of $\{f_i\}$, also called $\{f_i\}$, so that for each n $\{f_i|M_n\}$ converges to some $f^{(n)}$ as above. Define $f_o : M \to N$ by putting f_o equal to $f^{(n)}$ on M_n. f_o is well-defined since $f^{(n)}$ and $f^{(n+1)}$ agree a.e on M_n by pointwise convergence. Further, $f_o \in L^2_{1,loc}(M,N)$ and indeed $f_o \in H_\phi$ by (1) above. Lastly,

$$\begin{aligned} I \le E(f_o) &= \lim_{n\to\infty} E(f_o|M_n) \\ &= \lim_{n\to\infty} E(f^{(n)}) \\ &\le \lim_{n\to\infty} (\liminf_{i\to\infty} E(f_i|M_n)) \\ &\le \liminf_{i\to\infty} E(f_i) = I \end{aligned}$$

so that f_o is indeed the required map. □

4. *Regularity*

Let M be compact manifold (possibly with boundary).

Definition A map $f \in L^2_1(M,N)$ is said to be E-minimising on ε-balls if $E(f) \leq E(w)$ for any $w \in L^2_1(M,N)$ which agrees with f off some ball B of radius less than ε i.e. if $f = w$ on $M \backslash B$ and $(f-w)|B \in L^2_{1,0}(B,N)$.

Definition A C^∞ map $u : S^m \to N$ is said to be a *minimising tangent map* (MTM) if its homogeneous extension $\bar{u}$ to $\mathbb{R}^{m+1} \backslash \{0\}$ given by

$$\bar{u}(x) = u\left(\frac{x}{|x|}\right)$$

is E-minimising on compacta in $\mathbb{R}^{m+1} \backslash \{0\}$.

It is clear that minimising tangent maps are harmonic. We recall the fundamental regularity theorem of Schoen-Uhlenbeck [64], (see also Giaquinta-Giusti [32,33]):

Theorem 4.1 [64] Let $f \in L^2_1(M,N)$ be E-minimising on ε-balls for some ε. Then if N is compact:

(a) f is smooth in the interior of M off a set of Hausdorff dimension no greater than dimM-3.

(b) if there are no non-trivial MTMs of r-spheres into N for $2 \leq r \leq \dim M-1$ then f is a harmonic map smooth on the interior of M.

This theorem is a generalisation of the following theorem of Morrey which we shall need in Section 6.

Theorem 4.2 [52] If dimM=2 and N is homogeneously regular and $f \in L^2_1(M,N)$ is E-minimising on ε-balls for some ε, then f is a harmonic map smooth on the interior of M.

Lemma 4.3 The limit map of proposition 3.1, f_o, is E-minimising on ε-balls for some ε on each M_n.

Proof We choose ε to be less than half the 'width' of the tubular neighbourhoods about the generating curves of the various $\pi_1(M_n)$. Then, if $w \in L^2_1(M_n,N)$ and agrees with $f_o|M_n$ off some ε-ball, we can extend w to $\bar{w} \in L^2_{1,loc}(M,N)$ by setting $\bar{w}$ equal to f_o on $M\backslash M_n$ and it is clear that $\bar{w} \in \mathcal{H}_\phi$.

Thus

$$E(f_o|M_n) + E(f_o|M\backslash M_n) = E(f_o) \le E(\bar{w}) = E(w) + E(f_o|M\backslash M_n)$$

and the lemma follows. □

Theorem 4.4 Let M be a complete manifold and N a compact manifold admitting no non-trivial MTMs of r-spheres for $2 \le r \le \dim M-1$. Let $\phi \in C^\infty(M,N)$ with finite energy and define $\mathcal{H}_\phi$ as above. Then there exists a smooth harmonic map $f_o \in \mathcal{H}_\phi$ minimising energy among all maps in $\mathcal{H}_\phi$.

5. *Topology*

It is well-known that if N is a $K(\pi,1)$, the homotopy classes of maps from a compact M into N are in bijective correspondence with the conjugacy classes of homomorphisms from $\pi_1(M)$ to $\pi_1(N)$ (see eg. Spanier [71]). Thus, in that case, the continuous elements of $\mathcal{H}_\phi$ are all homotopic to ϕ on compacta.

Further we have the following theorem taught us by V.L. Hansen [39]:

Theorem 5.1 Let M,N be connected C-W complexes with M countable and N a $K(\pi,1)$. Let $f,g : M \to N$ be maps that are homotopic on compacta, then f,g are homotopic as maps from M into N.

Thus we may conclude

Theorem 5.2 Let M be complete Riemannian manifold and N a compact Riemannian manifold which is a $K(\pi,1)$, and admits no non-trivial MTMs of r-spheres for $2 \le r \le \dim M - 1$. Then any homotopy class of maps of M into N containing a map of finite energy, contains a smooth harmonic map minimising energy in the homotopy class.

Remark In the case of two-dimensional M, the topological condition on N can be weakened to only requiring that $\pi_2(N)$ vanishes. Also the 'MTM' condition is vacuous and so our theorem reduces to that of Lemaire in this case (see [49]).

The results of Sacks-Uhlenbeck include the following:

Proposition 5.3 [60] If N admits no non-trivial harmonic maps of 2-spheres, then the universal cover of N is contractible and N is a $K(\pi,1)$.

Thus compact manifolds N which admit no harmonic spheres satisfy the hypotheses of theorem 5.2. We note the following examples of such manifolds (see [20]).

(i) N has all sectional curvatures non-positive [7], or, more generally,

(ii) the universal cover of N has no focal points [18,76].

(iii) N is a surface whose universal cover has no conjugate points [12].

All these examples are subsumed under the general glass of manifolds whose universal covers are convex supporting, [36].

6. *Harmonic maps into non-compact manifolds*

An examination of the proof of theorem 5.2 shows that the compactness of N was used to ensure L^2_1-boundedness of the minimising sequence in proposition 3.1 and as a hypothesis in the regularity theorem 4.1. But if M is a surface, the regularity requirement can be weakened as in theorem 4.2 to the demand that N be homogeneously regular. So, in this case, we can prove existence theorems for non-compact range manifolds under conditions that ensure the L^2_1 boundedness of a minimising sequence.

Lemaire has discovered two such conditions, the first of which is a growth condition analogous to that of Uhlenbeck [72]:

> There exists a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{r\to\infty} g(r) = \infty$ such that for some $b \in N$, the geodesic ball $B(y, g(\mathrm{dist}(b,y)))$ is contractible for every $y \in N$.

Theorem 6.1 Let M be a Riemann surface and N a complete homogeneously regular manifold with $\pi_2(N) = 0$, which satisfies the above growth condition. Then any homotopy class of maps from M into N containing a map of finite energy, contains a smooth harmonic map minimising energy in the homotopy class.

Proof We may assume that M is complete since M is conformally equivalent to a complete Riemann surface and the energy is conformally invariant.

Further we assume the homotopy class is non-trivial, otherwise there is nothing to prove. Let ϕ be a smooth map in the homotopy class with finite energy. Since $\pi_2(N) = 0$ and ϕ is homotopically non-trivial, there is an element, $[\gamma]$ say, of $\pi_1(M)$ such that $\phi(\gamma)$ is not contractible. Now exhaust M by an increasing sequence $\{M_n\}$ of compact manifolds with boundary such that the image of γ is contained in M_1 and define $\mathcal{H}_\phi$ as before. We may assume that γ is one of the generators of each $\pi_1(M_n)$ and then, if $\{f_i\}$ is a minimising sequence for the energy in $\mathcal{H}_\phi$, we see that $f_i([\gamma])$ is homotopically non-trivial for all i.

As remarked above, it suffices to find a subsequence $\{f_j\}$ such that $\{f_j|M_n\}$ is uniformly bounded in $L_1^2(M_n, N)$, for each n.

By Fatou's Lemma, as in proposition 3.1, for almost all $s \in I$ there is a number K_s such that

$$\int_{S_1} |d(f_j \circ \gamma^s)|^2 dt \leq K_s,$$

for some subsequence $\{f_j\}$, and thus the lengths of the curves $f_j(\gamma^s)$ are uniformly bounded by $(|S^1|K_s)^{\frac{1}{2}}$.* So $\{f_j\}$ is uniformly L_1^2-bounded on γ^s and therefore, by a result of Morrey (Lemma 9.4.14 [52]), $\{f_j|M_n\}$ is uniformly bounded in $L_1^2(M_n, N)$ for each n, whence the theorem follows. □

Even when the growth condition is not satisfied it is possible to prove existence of harmonic maps in some homotopy classes.

Theorem 6.2 Let M be a Riemann surface and N a complete homogeneously regular manifold with $\pi_2(N) = 0$. Let $\mathcal{H}$ be a homotopy class of maps from M to N containing a map of finite energy and suppose there is a loop α in M and a compact subset K of N such that for all $f \in \mathcal{H}$, $f(\alpha) \cap K$ is non-empty. Then $\mathcal{H}$ contains a harmonic map minimising energy in $\mathcal{H}$.

* The growth condition now ensures that these curves must stay within a fixed distance of b otherwise they would end up in some contractible ball.

Remark This last condition (again due to Lemaire) is implied by the following condition of Schoen-Yau [67]:

There exists $f \in \mathcal{H}$, a cycle $\alpha \in H_1(M)$ and a compact subset K of N such that $f_*(\alpha)$ is not homologous to zero in $H_1(N, N\backslash K)$.

Example The above condition is easily seen to be satisfied by any non-trivial homotopy class of maps into a countable connected sum of isometric 2-tori.

Proof of Theorem 6.2 The argument in Theorem 6.1 applied to α shows that the curves $\{f_j(\alpha)\}$ stay within a fixed distance of K for some subsequence $\{f_j\}$ and then we proceed as before. $\square$

CHAPTER 3

MAPS THAT COMMUTE WITH THE CO-DIFFERENTIAL

In this chapter, we characterise those surjective maps from one Riemannian manifold onto another whose action on vector-bundle valued differential forms commutes with the co-differential, d^*.

This extends and corrects a theorem of Watson [73] who considered this problem for real-valued differential forms.

We remark that since all maps commute with exterior derivatives, d^*-commuting maps commute with the harmonic map equation and the Yang-Mills equations.

1. *Definitions and Notations*

Let (M,g) be a Riemannian manifold and $\pi: E \to M$ a vector bundle with a Riemannian structure ie. a metric a and a connection ∇ with $\nabla a = 0$.

The E-valued p-forms on M are defined to be the sections of $\Lambda^p T^*M \otimes E$.

The co-differential $d^*: C^\infty(\Lambda^p T^*M \otimes E) \to C^\infty(\Lambda^{p-1} T^*M \otimes E)$ is the formal adjoint (with respect to $g \otimes a$) of the exterior derivative. It can be defined in terms of the Levi-Civita connection on M and the connection on E as follows:

$d^*\alpha_x(X_1,..,X_{p-1}) = -\sum_i (\nabla_{E_i}\alpha)(E_i,X_1,..,X_{p-1})$ where $x \in M$, $\{X_i\} \in T_xM$, $\{E_i\}$ is an orthonormal basis of T_xM, $\alpha \in C^\infty(\Lambda^p T^*M \otimes E)$ and ∇ is the induced connection on $\Lambda^p T^*M \otimes E$.

Now let (M,g), (N,h) be Riemannian manifolds of dimension m and n respectively and $\pi: E \to N$ a vector bundle over N with a Riemannian structure. Let $\phi: M \to N$ be a surjective map. We say that ϕ is *d*-commuting on E-valued p-forms* if, for all $\alpha \in C^\infty(\Lambda^p T^*M \otimes E)$,

$d^*\phi^*\alpha = \phi^* d^*\alpha$ where both sides are ϕ^{-1} E-valued p-1 forms

and $\phi^{-1}E$ is equipped with the pull-back connection.

If ϕ is a submersion, we call the kernel of $d\phi$ the *vertical distribution* and its orthogonal complement with respect to g the *horizontal distribution*.

If $d\phi$ is an isometry on the horizontal distribution, then ϕ is called a *Riemannian submersion*.

2. *The Characterisation*

Theorem 2.1 (i) $\phi: M \to N$ is d*-commuting on E-valued 1-forms if and only if ϕ is a harmonic Riemannian submersion.

(ii) For any $2 \le p \le n$, $\phi: M \to N$ is d*-commuting on E-valued p-forms if and only if ϕ is a harmonic Riemannian submersion with integrable horizontal distribution.

Remark In the case of real-valued 1-forms, theorem 2.1(i) is due to Watson [73].

The proof of the theorem is contained in the following two lemmata, the first of which is due to Watson [73] in the case of real-valued forms.

Lemma 2.2 Let $1 \le p \le n$, then ϕ is d*-commuting on E-valued p-forms if and only if ϕ is a Riemannian submersion and $d^*\Lambda^p d\phi = 0$, where $\Lambda^p d\phi$ is thought of as a $\Lambda^p \phi^{-1}TN$-valued p-form.

Proof Suppose ϕ is d*-commuting on E-valued p-forms for some p. Fix $x_o \in M$ and let $\{x_1,\ldots,x_m\}$, $\{y_1,\ldots,y_n\}$ be local co-ordinates about x_o and $\phi(x_o)$ respectively and $\{\sigma_\alpha\}$ a local trivialisation of E about $\phi(x_o)$.

Let ω be an E-valued p-form on N vanishing at $\phi(x_o)$, so that locally

$$\omega = \omega^\alpha_{a_1\ldots a_p} dy^{a_1}\wedge\ldots\wedge dy^{a_p} \otimes \sigma_\alpha$$

with each $\omega^\alpha_{a_1}\ldots_{a_p}$ vanishing at $\phi(x_o)$. Thus at x_o,

$$\begin{aligned}
-(d^*\phi^*\omega)^\alpha_{i_1\ldots i_{p-1}} &= g^{ij}(\nabla_{\partial_i}\phi^*\omega)^\alpha_{j,i_1\ldots i_{p-1}} \\
&= g^{ij}\partial_i(\phi^*\omega^\alpha_{j,i\ \ldots i_{p-1}}) \\
&= g^{ij}\partial_i(\omega^\alpha_{a,a_2\ldots a_p}(\phi).\ \phi^a_j\phi^{a_2}_i\ldots\phi^{a_p}_{i_{p-1}}) \\
&= \partial_b\omega^\alpha_{a,a_2\ldots a_p}(\phi^b_i\phi^a_j g^{ij})\phi^{a_2}_{i_1}\ldots\phi^{a_p}_{i_{p-1}}\ .
\end{aligned}$$

Similarly, at x_o;

$$\begin{aligned}
-(\phi^*d^*\omega)^\alpha_{i_1\ldots i_{p-1}} &= \phi^{a_2}_{i_1}\ldots\phi^{a_p}_{i_{p-1}}(-d^*\omega)^\alpha_{a_2\ldots a_p} \\
&= \phi^{a_2}_{i_1}\ldots\phi^{a_p}_{i_{p-1}}\ h^{ab}(\phi)(\nabla_{\partial_b}\omega)^\alpha_{a,a_2,,,a_p} \\
&= \phi^{a_2}_{i_1}\ldots\phi^{a_p}_{i_{p-1}}\ h^{ab}(\phi)\partial_b\omega^\alpha_{a,a_2\ldots a_p}\ .
\end{aligned}$$

Now $\partial_b \omega^\alpha_{a,a_2 \ldots a_p}(\phi)$ is arbitrary, so that if rank $d\phi(x_o)$ is greater than $(p-1)$ we can find $\{a_2,\ldots,a_p\}$, $\{i_1,\ldots,i_{p-1}\}$ such that $\phi^{a_2}_{i_1}\ldots\phi^{a_p}_{i_{p-1}}(x_o) \neq 0$ whence

$$h^{ab}(\phi(x_o)) = g^{ij}\phi^b_i\phi^a_j(x_o)$$

which is precisely the condition that ϕ be a Riemannian submersion at x_o. Thus, for any $x \in M$, ϕ is either a Riemannian submersion at x or has rank $\leq (p-2)$ at x. We shall dispose of the second possibility below.

We have the following identity, valid for any map $\phi : M \to N$ and any E-valued p-form on N, which I learnt from Dr. John Rawnsley:

$$\nabla(\phi^*\omega) = \omega \circ \nabla \Lambda^p d\phi + \phi^*(\nabla\omega).$$

This is just a Leibnitz formula when we identify $\phi^*\omega$ with $\phi^{-1}\omega \circ \Lambda^p d\phi$ where $\phi^{-1}\omega$ is a section of $\phi^{-1}(\Lambda^p T^*N \otimes E) = \phi^{-1}\Lambda^p T^*N \otimes \phi^{-1}E$ and $\Lambda^p d\phi$ is viewed as a section of $\mathrm{Hom}(\Lambda^p TM, \Lambda^p \phi^{-1}TN)$.

Now, if ϕ is a Riemannian submersion at x, taking traces, we have

$$d^*\phi^*\omega = \omega(d^*\Lambda^p d\phi) + \phi^* d^*\omega \text{ at } x.$$

Since ϕ is d^*-commuting, we conclude that $d^*\Lambda^p d\phi(x) = 0$. If $d\phi(x)$ has rank $\leq (p-2)$, then $\phi^*(\nabla\omega)$ and $\phi^* d^*\omega$ vanish identically and, again taking traces, we have

$$d^*\phi^*\omega = \omega(d^*\Lambda^p d\phi) + \phi^* d^*\omega \text{ at } x.$$

Again we conclude that $d^*\Lambda^p d\phi = 0$ at x and thus that $d^*\Lambda^p d\phi$ vanishes identically. Since $d\Lambda^p d\phi$ always vanishes we may apply the unique continuation theorem of Aronszajn-Krzywicki-Szarski [4] to conclude that since ϕ is surjective, rank $d\phi \geq p$ on a dense open set. Thus ϕ is a Riemannian submersion on a dense open set and so is a Riemannian submersion everywhere.*

The converse statement is immediate from the above formulas. □

Part (i) of the theorem is now proved since the vanishing of $d^*d\phi$ is precisely the condition that ϕ be harmonic.

Lemma 2.3 If $\phi: M \to N$ is a Riemannian submersion, then, for $2 \leq p \leq n$, $d^*\Lambda^p d\phi$ vanishes if and only if ϕ is harmonic with integrable horizontal distribution.

Proof Let $E_1, \ldots, E_m$ be a local orthonormal frame field about $x \in M$, with $E_1, \ldots, E_n$ spanning the horizontal distribution and $E_{n+1}, \ldots, E_m$ spanning the vertical distribution. Let $X_1, \ldots, X_{p-1} \in T_xM$. Then

$$d^*\Lambda^p d\phi(X_1, \ldots, X_{p-1}) = -\sum_{i \leq m} \nabla_{E_i}(\Lambda^p d\phi)(E_i, X_1, \ldots, X_{p-1})$$

$$= -\sum_{i \leq n} \sum_{k \leq p-1} d\phi(E_i) \wedge d\phi(X_1) \wedge \ldots \wedge d\phi(X_{k-1}) \wedge \nabla_{E_i}(d\phi)(X_k) \wedge \ldots \wedge d\phi(X_{p-1})$$

$$-\sum_i \nabla_{E_i}(d\phi)(E_i) \wedge d\phi(X_1) \wedge \ldots \wedge d\phi(X_{p-1}).$$

From Hermann [40], since ϕ is a Riemannian submersion, we have that $\nabla_X(d\phi)Y = 0$ for all horizontal X, Y and so if $X_1, \ldots, X_{p-1}$ are all horizontal then

$$d^*\Lambda^p d\phi(X_1, \ldots, X_{p-1}) = d^*d\phi \wedge d\phi(X_1) \wedge \ldots \wedge d\phi(X_{p-1}).$$

* since the set on which ϕ is a Riemannian submersion is necessarily closed.

Thus $d^*\Lambda^p d\phi$ vanishes on horizontal vectors if and only if ϕ is harmonic, since $d\phi$ is surjective.

If X_1 is vertical,

$$d^*\Lambda^p d\phi(X_1,\ldots,X_{p-1}) = -\sum_{i\le n} d\phi(E_i)\wedge\nabla_{E_i}(d\phi)(X_1)\wedge d\phi(X_2)\wedge\ldots\wedge d\phi(X_{p-1})$$

which vanishes if and only if

$$\sum_{i\le n} d\phi(E_i)\wedge\nabla_{E_i}(d\phi)(X_1) = 0.$$

An application of Cartan's Lemma shows that this is equivalent to

$$h(d\phi(E_i),\nabla_{E_j}(d\phi)(X_1)) = h(d\phi(E_j),\nabla_{E_i}(d\phi)(X_1)) \text{ for } i,j\le n,$$

or, since ϕ is a Riemannian submersion;

$$g(E_i,\nabla_{E_j}X_1) = g(E_j,\nabla_{E_i}X_1) \text{ for } i,j\le n.$$

This last is clearly equivalent to the integrability of the horizontal distribution and the lemma is proved. □

3. *Comments and a Counter-example*

(i) Eells and Sampson [27] have shown that a Riemannian submersion is harmonic if and only if its fibres are minimal.

(ii) Goldberg and Ishihara [35] have shown that the harmonic Riemannian submersions with integrable horizontal distribution are precisely the maps that commute with the Laplacian on p-forms, for $p\ge 2$.

(iii) Using Hodge Theory and the De Rham Isomorphism Theorem, inequalities between the Betti numbers of manifolds admitting d*-commuting maps can be derived, see Watson [73].

In [73], Watson asserted that the d*-commuting maps for p-forms, for $p \geq 2$, were only the totally geodesic Riemannian submersions ie. Riemannian submersions with totally geodesic fibres and integrable horizontal distribution. Some of the main steps of his proof are incorrect (see eg. his lemma 3.6) and we now present a counter-example to his assertion viz. a Riemannian submersion with integrable horizontal distribution and minimal, but not totally, geodesic fibres.

Let (M,g) be an arbitrary Riemannian manifold and $\phi:M \to \mathbb{R}$ a non-constant function. Consider the family of flat metrics on the 2-torus T^2 given by

$$h_u = \begin{pmatrix} e^u & 0 \\ 0 & e^{-u} \end{pmatrix}, \quad u \in \mathbb{R}.$$

Equip $M \times T^2$ with the metric $g(x) + h_{\phi(x)}$ at $(x,y) \in M \times T^2$, then projection onto M is a Riemannian submersion with integrable horizontal distribution. The following lemma will show that the fibres of this map are minimal but not totally geodesic.

Lemma 3.1 Let $\phi:(M,g) \to (N,h)$ be a Riemannian submersion. Let g' denote the fibre metric and v_g' the fibre volume element (ie. compose the metric and volume element with the projection onto the vertical distribution). Then,

(i) the fibres of ϕ are totally geodesic if and only if $L_X g'$ vanishes on the vertical distribution for all horizontal vector fields X.

(ii) the fibres of ϕ are minimal if and only if $L_X v_g'$ vanishes on the vertical distribution for all horizontal vector fields X.

Thus the fibres are totally geodesic if and only if "the fibre metric is independent of the fibre" and minimal if and only if "the fibre volume element is independent of the fibre".

Proof (i) Let V_1, V_2 be vertical and X a horizontal vector field. Then

$$L_X g'(V_1,V_2) = Xg(V_1,V_2) - g(L_X V_1,V_2) - g(V_1,L_X V_2)$$

$$= Xg(V_1,V_2) - g(\nabla_X V_1,V_2) - g(V_1,\nabla_X V_2) + g(\nabla_{V_1} X,V_2) + g(\nabla_{V_2} X,V_1)$$

$$= -2g(\nabla_{V_1} V_2,X) = -2g(\beta(V_1,V_2),X),$$

where β is the second fundamental form of the fibres.

(ii) Let $E_1,\ldots,E_{m-n}$ be a local orthonormal basis for the vertical distribution so that $v_g'(E_1,\ldots,E_{m-n}) = 1$ and let X be horizontal.

$$(L_X v_g')(E_1,\ldots,E_{m-n}) = Xv_g'(E_1,\ldots,E_{m-n}) - \Sigma_{i<m-n} v_g'(E_1,..L_X E_i..E_{m-n})$$

$$= -\Sigma_{i<m-n} g(L_X E_i,E_i)$$

$$= -\Sigma_i g(\nabla_X E_i,E_i) + \Sigma_i g(\nabla_{E_i} X,E_i)$$

$$= -\Sigma_i g(\nabla_{E_i} E_i,X) = -\, g(\mathrm{Tr}\beta,X). \qquad \square$$

CHAPTER 4

UNIQUE CONTINUATION PROPERTIES OF HARMONIC AND HOLOMORPHIC MAPS

A solution of a second order elliptic partial differential equation cannot have a zero of infinite order without vanishing identically. This is called the (strong) unique continuation property. The weak unique continuation property is possessed by solutions that cannot vanish on an open set without vanishing identically.

In this chapter, we examine some unique continuation type properties of harmonic and holomorphic maps and harmonic vector bundle valued forms.

A. *The Aronszajn-Cordes Theorem and Related Results*

1. *Introduction*

The main source of unique continuation properties in elliptic analysis are the theorems of Aronszajn [3], Cordes [17] and Aronszajn-Kryzwicki-Szarski [4]:

Theorem 1.1 [3][17] Let $u^1 \ldots u^m$ be C^2 functions on a connected open subset D of $\mathbb{R}^n$ and L a second order elliptic linear differential operator whose principal part has C^2 coefficients on D. Suppose there exists a positive constant M such that for each α: $1 \le \alpha \le m$

$$|Lu^\alpha(x)|^2 \le M \left\{ \sum_\beta |u^\beta(x)|^2 + \sum_\beta |\nabla u^\beta(x)|^2 \right\} \quad \text{all} \quad x \in D .$$

Then, if all u^α vanish on a non-empty open set, each u^α vanishes identically on D.

Theorem 1.2 [4] Let M be a connected Riemannian manifold with $C^{0,1}$ metric and let $w^1 \dots w^m$ be differential p-forms with coefficients in $L^2_{1,loc}$. Suppose that for every compact K in M, there is a positive constant M_K such that for each α: $1 \le \alpha \le m$

$$|dw^\alpha(x)|^2 + |d{*}w^\alpha(x)|^2 \le M_K \sum_\beta |w^\beta(x)|^2 \qquad \text{a.e. } x \in K .$$

Then, if all w^α vanish almost everywhere on a non-empty open subset of M, each w^α vanishes almost everywhere on M.

Remark In fact, in both of the above theorems, we need only require that each u^α (w^α) has a zero of infinite order in 1-mean at some point x_0:

$$\int_{B_r(x_0)} |u^\alpha| = O(r^{n+p}) \quad \text{all} \quad p > 0 \tag{1}$$

For theorem 1.2, this means that the coefficients of w^α satisfy (1) in one (and hence every) co-ordinate patch about x_0.

2. *Applications*

We first extend theorem 1.2 to the case of differential forms with values in a Riemann connected vector bundle.

Henceforth all ingredients will be assumed smooth (C^∞) unless otherwise stated.

Theorem 2.1 Let M be a connected n-dimensional Riemannian manifold and $\pi: E \to M$ a Riemann connected vector bundle with connection ∇. Let $w^1, \dots, w^m$ be E-valued p-forms on M with coefficients in $L^2_{1,loc}$. Suppose that for each compact K in M there exists a positive constant M_K such that for each α: $1 \le \alpha \le m$

$$|dw^{\alpha}(x)|^2 + |d^*w^{\alpha}(x)|^2 \le M_K \sum_{\beta} |w^{\beta}(x)|^2 \quad \text{a.e.} \quad x \in K \, . \tag{2}$$

Then, if all w^{α} vanish almost everywhere on some non-empty open set, each w^{α} vanishes almost everywhere on M.

Proof Let Ω denote the largest open set on which all w^{α} vanish a.e. and let $x_0 \in b\Omega$. Let $e_1 \dots e_r$ be a local orthonormal basis for E over a neighbourhood U of x_0 and let K be a compact in U.

Let $w^{\alpha} = w_i^{\alpha} \otimes e_i$ on U where each w_i^{α} is a real valued p-form. Then $dw^{\alpha} = dw_i^{\alpha} \otimes e_i + (-1)^{p+1} w_i^{\alpha} \wedge \nabla e_i$ and writing

$d^* = (-1)^{np+n+1} *d*$, where $*$ is the Hodge $*$ operator we have

$$d^*w = d^*w_i^{\alpha} \otimes e_i + (-1)^{np+n+p} *(* w_i^{\alpha} \wedge \nabla e_i) \, .$$

Almost everywhere on K we have

$|dw^{\alpha}|^2 + |d^*w_{\alpha}|^2 \le M_K \sum_{\beta} |w_{\beta}|^2$ whence

$$|dw_i^{\alpha} \otimes e_i + (-1)^{p+1} w_i^{\alpha} \wedge \nabla e_i|^2 + |d^*w_i^{\alpha} \otimes e_i + (-1)^{np+n+p} *(* w_i^{\alpha} \wedge \nabla e_i)|^2 \le M_K \sum_{\beta} |w_i^{\beta}|^2 .$$

Using the Cauchy-Schwartz inequality we have

$$|dw_i^{\alpha}|^2 + |d^*w_i^{\alpha}|^2 \le \text{const} \left\{ \sum_{j,\beta} \left(|w_j^{\beta}|^2 + |w_j^{\beta} \wedge \nabla e_j|^2 + |*w_j^{\beta} \wedge \nabla e_j|^2 \right) \right\}$$

and the last two terms on the right are now estimated in terms of $|w_j^{\beta}|^2$ and derivatives of the $\{e_i\}$, bounded on K, since for u a 1-form and v a p-form we have

$|u \wedge v|^2 \le |u|^2 |v|^2$ (see [4]).

Thus we have, for each α, i

$|dw_i^{\alpha}|^2 + |d^*w_i^{\alpha}|^2 \le M'_K \sum_{\beta, j} |w_j^{\beta}|^2$ and applying theorem 2.1 to w_i^{α} on U shows that $x_0 \in \Omega$ and the theorem follows from the connectedness of M.

□

Corollary 2.2 Let M,E be as above and let $w_1 \ldots w_m$ be E-valued p-forms on M satisfying inequality (2). Let $S \subset M$ be a hypersurface. If each w_α vanishes on S, then each w_α vanishes identically.

Proof In some co-ordinate neighbourhood $(U, x_1 \ldots x_n)$ assume that S is given by $\{x: x_n = 0\}$. Define $\tilde{w}_\alpha$ by $\tilde{w}_\alpha = w_\alpha$ on $x_n \leq 0$

$\tilde{w}_\alpha = 0$ on $x_n \geq 0$. Then each $\tilde{w}_\alpha$ is L_1^2 on U and satisfies inequality (2) so that by theorem 2.1 each $\tilde{w}_\alpha \equiv 0$. Thus each w_α vanishes for $x_n \leq 0$ and a further application of theorem 2.1 shows that each w_α vanishes identically. □

We now consider some examples to which the above theory may be applied.

a) *Harmonic maps* Let $\phi: M \to N$ be a map of Riemannian manifolds. Then $d\phi$ is a $\phi^{-1}TN$-valued 1-form and $dd\phi = 0$. Further $d^*d\phi = 0$ if and only if ϕ is harmonic so we have:

Proposition 2.3 Let $\phi: M \to N$ be a harmonic map where M is connected.

i) If ϕ is constant on a non-empty open set, ϕ is constant on M.

ii) If ϕ is constant on some hypersurface $S \subset M$ and has vanishing normal derivative to S, then ϕ is constant on M.

Remark Part (i) of the proposition is due to Sampson [62].

b) *f-holomorphic maps* An f-structure on a Riemannian manifold M is a skew-symmetric section F of $C^\infty(\mathrm{End}(TM))$ of constant rank, such that

$$F^3 = -F.$$

The best known examples are almost complex structures of almost Hermitian manifolds and by analogy with that case we say that a map $\phi: M \to N$ of manifolds with f-structures F^M, F^N is *f-holomorphic* if

$$F^N \circ d\phi = d\phi \circ F^M .$$

Proposition 2.4 Let M be a connected Hermitian manifold with almost complex structure J and N a manifold with f-structure F. Let $\phi: M \to N$ be f-holomorphic. Then

i) if ϕ is constant on a non-empty open set, ϕ is constant on M.

ii) if ϕ is constant on some hypersurface $S \subset M$ and has vanishing normal derivative to S, then ϕ is constant on M.

The theorem is also true for f-holomorphic maps from a manifold with f-structure *into* an almost Hermitian manifold.

Proof It suffices to establish inequality (2) for $d\phi$.

Since $d\phi \circ J = F \circ d\phi$ we have $d\phi = -F \circ d\phi \circ J$. Thus

$d^* d\phi = -$ Trace $\nabla d\phi =$ Trace $\nabla(F \circ d\phi \circ J)$.

Now

$$\nabla(F \circ d\phi \circ J) = \nabla F \circ d\phi \circ J + F \circ \nabla d\phi \circ J + F \circ d\phi \circ \nabla J.$$

Let $E_1, \ldots, E_m, JE_1, \ldots, JE_m$ be a local orthonormal basis for TM, then

$$\text{Trace } F\nabla d\phi J = F\nabla d\phi(E_i, JE_i) + F\nabla d\phi(JE_i, J^2E_i) = 0$$

since $\nabla d\phi$ is symmetric and $J^2 = -\text{Id}$.

Thus

$$|d^* d\phi|^2 \leq \text{const } |d\phi|^2$$

where the constant depends on F, J and their derivatives. A similar argument is used for f-holomorphic maps *into* almost Hermitian manifolds.

□

Remark Using Theorem 1.1 and the fact that harmonic and f-holomorphic maps satisfy second order quasi-linear elliptic p.d.e.'s we may strengthen Propositions 2.3 and 2.4 as follows:

Under the hypotheses of Propsition 2.3 (2.4), if ϕ_1, ϕ_2 are harmonic (f-holomorphic) and agree either on a non-empty open set or agree up to 1-jets on some hypersurface then $\phi_1 \equiv \phi_2$ on M.

C. *Yang-Mills fields* (see Atiyah [5])

Let G be a compact Lie group with Lie algebra g and $P \to M$ a principal G-bundle over a Riemannian manifold M. A connection α on P has a curvature F^α which can be thought of as a 2-form on M with values on adP, the vector bundle associated with P by the adjoint representation on g. Now if we equip adP with a G-invariant metric, α induces a connection ∇^α on adP so that adP is Riemann-connected.

A *Yang-Mills connection* α is one whose curvature satisfies

i) $dF^\alpha = 0$

ii) $d^*F^\alpha = 0$ when adP has connection ∇^α.

We note that (i) is just the Bianchi identity satisfied by any curvature 2-form. The curvature of a Yang-Mills connection is called a *Yang-Mills field*.

We say that α is *flat* at $x \in M$ if $F^\alpha(x) = 0$. If $\dim M = 4$, then the Hodge star operator $*$ sends 2-forms to 2-forms and $*^2 = \mathrm{Id}$. We say that α is *self-dual* if $F^\alpha = *F^\alpha$ and *anti-self-dual* if $F^\alpha = -*F^\alpha$. It is clear that (anti)self-dual connections are Yang-Mills.

We have

Proposition 2.5 $P \to M$ be a principal G-bundle over a connected Riemannian manifold M and let α be a Yang-Mills connection. If α is flat on a non-empty open set or along a hypersurface then α is flat on M. If $\dim M = 4$ and α is (anti)self-dual on a non-empty set or along a hypersurface then α is (anti)self-dual on M.

Proof We remark that $d^* = *d^*$ on 2 forms, so $d(F^\alpha \pm *F^\alpha) = d^*(F^\alpha \pm *F^\alpha) = 0$. □

Lastly, we generalise a result of Siu [69] concerning holomorphic maps of Kahler manifolds.

Definitions An (almost) Hermitian manifold is said to be *(almost) co-symplectic* if the Kahler form of the manifold is co-closed.

Let N be a Riemannian manifold with f-structure F. Then the complexified tangent space of N splits into eigenspaces of F with eigenvalues $+i$, $-i$, 0 denoted T^+N, T^-N, T^0N respectively.

(N,F) is said to satisfy condition 'A' [59] if

$\nabla_X(C^\infty(T^+N)) \subset C^\infty(T^+N)$ for $X \in T^-N$.

Remark If N is almost Hermitian (i.e. if $F^2 = -\mathrm{Id}$) then condition 'A' reduces to the familar condition that $d\omega^{N(1,2)} = 0$ where ω^N is the Kahler form of N.

Theorem 2.6 Let (M,g,J) be a connected co-symplectic hermitian manifold and (N,h,F) a Riemannian manifold with f-structure satisfying condition 'A' Let $\phi: M \to N$ be a harmonic map and suppose that ϕ is f-holomorphic on some non-empty open set. Then ϕ is f-holomorphic on M.

Remark In the case, where M,N are Kahler manifolds this result is due to Siu [69].

Proof Let Ω be the largest open set on which ϕ is f-holomorphic and let $x_o \in b\Omega$. Let $(U, z_1 \dots z_m)$ be holomorphic co-ordinates about x_o and $e_1, \dots, e_r, e_{\bar{1}}, \dots, e_{\bar{r}}, e_{r+1}, \dots, e_n$ be a local basis for $TN^{\mathbb{C}}$ about $\phi(x_o)$ with $\{e_\alpha\}_1^r$ spanning T^+N, $\{e_{\bar{\alpha}} = \overline{e_\alpha}\}_1^r$, spanning T^-N and $\{e_\alpha = \overline{e_\alpha}\}_{r+1}^n$ spanning T^oN.

Since ϕ is harmonic, in local co-ordinates we have

$$\tau_\phi{}^\alpha = g^{i\bar{j}} \frac{\partial^2 \phi^\alpha}{\partial z_i \partial \bar{z}_j} + g^{i\bar{j}} \Gamma^\alpha_{BC} \phi^B_i \phi^C_{\bar{j}} = 0 \qquad 1 \le \alpha \le n .$$

Differentiating both sides by $\dfrac{\partial}{\partial \bar{z}_k}$ we have

$$g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \phi^\alpha_{\bar{k}} + \frac{\partial}{\partial \bar{z}_k} g^{i\bar{j}} \frac{\partial}{\partial z_i} \phi^\alpha_{\bar{j}} + \frac{\partial}{\partial \bar{z}_k} \left(g^{i\bar{j}} \Gamma^\alpha_{BC} \phi^B_i \phi^C_{\bar{j}} \right) = 0.$$

Now condition 'A' implies $\Gamma^\gamma_{\alpha\bar{\beta}} = 0 \quad 1 \le \alpha, \beta \le r, \quad 1 \le \gamma \le n$, so that

$$\left| g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \phi^\alpha_{\bar{k}} \right|^2 \le M \sum_{\beta, j} \left\{ \left| \phi^\beta_{\bar{j}} \right|^2 + \left| \nabla \phi^\beta_{\bar{j}} \right|^2 \right\} \qquad , \text{ where M depends}$$

on the derivatives of ϕ, $g^{i\bar{j}}$ and Γ^α_{BC}. Now writing $\phi^\alpha_{\bar{k}}$ as $u^\alpha_k + iv^\alpha_k$ and using the fact that $L = g^{i\bar{j}} \dfrac{\partial^2}{\partial z_i \partial \bar{z}_j}$ is a real operator we have

$$|Lu^\alpha_k|^2 \le M \sum_{\beta, j} \left\{ |u^\beta_j|^2 + |v^\beta_j|^2 + |\nabla u^\beta_j|^2 + |\nabla v^\beta_j|^2 \right\}$$

$$|Lv^\alpha_k|^2 \le M \sum_{\beta, j} \left\{ |u^\beta_j|^2 + |v^\beta_j|^2 + |\nabla u^\beta_j|^2 + |\nabla v^\beta_j|^2 \right\} \qquad \text{so we may apply}$$

theorem 1.1 to the system $\left\{ u^\beta_j, v^\beta_j \right\}_{\beta, j}$ and the theorem follows.

□

B. Unique continuation on surfaces via holomorphicity

The study of harmonic maps of two-dimensional domains has many interesting features, one of which is the intimate relationship with holomorphic function theory provided by holomorphic differentials associated with harmonic maps (see [22]). In this section, we use this extra structure to prove unique continuation theorems for harmonic maps from surfaces without recourse to theorems A1.1 and A1.2.

1. *Holomorphic vector bundles on Riemann surfaces and their applications*

We recall a theorem of Kozsul-Malgrange [46],

Theorem 1.1 Let M be a Riemann surface and $\pi: E \to M$ a complex vector bundle with connection ∇. Then E admits a unique holomorphic structure such that a local section s of E is holomorphic iff $\nabla_{\frac{\partial}{\partial \bar{z}}} s = 0$.

Theorem 1.2 Let M be a connected Riemann surface and N a Riemannian manifold. Let $\phi: M \to N$ be a non-constant harmonic map. Then $d\phi$ has only discrete zeroes.

Proof Let $z = x + iy$ be an isothermal co-ordinate on M. Endow the pull-back of the complexified tangent bundle of N, $\phi^{-1}TN^C \to M$ with the holomorphic structure of theorem 1.1 associated with the pull-back of the Levi-Civita connection on N, ∇^ϕ. Then

$$\phi \text{ is harmonic if and only if } \qquad \nabla_{\frac{\partial}{\partial \bar{z}}} \phi_* \frac{\partial}{\partial z} = 0$$

so that $\phi_* \frac{\partial}{\partial z} = \frac{1}{2}\left(\phi_* \frac{\partial}{\partial x} - i\phi_* \frac{\partial}{\partial y}\right)$ is a local holomorphic section of $\phi^{-1}TN^C$ and the theorem follows. □

Theorem 1.3 Let M be a connected Riemann surface and N a Kahler manifold. Let $\phi: M \to N$ be a harmonic non $\pm$ holomorphic map. Then ϕ only is $\pm$ holomorphic on a discrete set of points.

Proof Decompose the complexified tangent bundle of TN into the eigenspaces of the complex structure $T^{(1,0)}N$, $T^{(0,1)}N$. Since N is Kahler, the Levi-Civita connection ∇ of N preserves these eigenspaces and thus $\phi^{-1}T^{(1,0)}N$, $\phi^{-1}T^{(0,1)}N$ have holomorphic structures associated with ∇^{ϕ} as above.

Let $d\phi = \partial\phi + \bar{\partial}\phi$

where $\partial\phi: TM^C \to \phi^{-1}T^{(1,0)}N$

$\bar{\partial}\phi: TM^C \to \phi^{-1}T^{(0,1)}N$ are defined by composition with the appropriate projections.

Then ϕ is harmonic iff $\nabla^{\phi}_{\frac{\partial}{\partial \bar{z}}} \partial\phi\left(\frac{\partial}{\partial z}\right) = 0$ iff $\nabla^{\phi}_{\frac{\partial}{\partial \bar{z}}} \bar{\partial}\phi\left(\frac{\partial}{\partial z}\right) = 0$ for $z = x + iy$ an isothermal co-ordinate on M. Thus

$$\partial\phi\left(\frac{\partial}{\partial z}\right),\ \bar{\partial}\phi\left(\frac{\partial}{\partial z}\right)$$

are local holomorphic sections of $\phi^{-1}T^{(1,0)}N$ and $\phi^{-1}T^{(0,1)}N$ respectively. The theorem follows by observing that ϕ is holomorphic iff $\bar{\partial}\phi\left(\frac{\partial}{\partial z}\right) = 0$ and anti-holomorphic iff $\partial\phi\left(\frac{\partial}{\partial z}\right) = 0$. □

2. *Holomorphic differentials*

Let M be a Riemann surface and (N,h) a Riemannian manifold. Let $\phi: M \to N$ be a smooth map and consider $\phi^*h^{(2,0)}$, the quadratic form given in local holomorphic co-ordinates by

$$\phi^*h^{(2,0)} = h\left(d\phi\left(\frac{\partial}{\partial z}\right),\ d\phi\left(\frac{\partial}{\partial z}\right)\right)dz^2 .$$

We have the well-known

Proposition 2.1 i) $\phi^*h^{(2,0)} \equiv 0$ iff ϕ is weakly conformal.

ii) if ϕ is harmonic, $\phi^*h^{(2,0)}$ is a holomorphic quadratic differential.

From proposition 2.1 we have immediately

Proposition 2.2 Let M be a connected Riemann surface and N a Riemannian manifold. Let ϕ: M$\to$N be a harmonic non-conformal map, then ϕ is only conformal on a discrete set of points.

A useful generalisation of conformality is *isotropy* which has played an important part in the classification theorems of Calabi [13] and Eells-Wood [29], (see also Chapter 7).

Definitions i) Let M be a Riemann surface and (N,h) a Riemannian manifold. A map ϕ: M$\to$N is real isotropic if

$$(\nabla^{\alpha}_{\frac{\partial}{\partial z}} \phi_*\frac{\partial}{\partial z}, \nabla^{\beta}_{\frac{\partial}{\partial z}} \phi_*\frac{\partial}{\partial z}) = 0$$

all $\alpha,\beta \geq 0$. Here (.,.) is the complex bilinear extension of h and $\nabla^{\alpha}_{\frac{\partial}{\partial z}} = \nabla_{\frac{\partial}{\partial z}} \circ \ldots \circ \nabla_{\frac{\partial}{\partial z}}$ α times with ∇ the pull-back connection on $\phi^{-1}TN$.

ii) [29] Now suppose that N is a Kahler manifold. A map ϕ: M$\to$N is complex isotropic if

$$<\nabla^{\alpha}_{\frac{\partial}{\partial z}} \partial\phi(\tfrac{\partial}{\partial z}) \quad \nabla^{\beta}_{\frac{\partial}{\partial \bar{z}}} \partial\phi(\tfrac{\partial}{\partial \bar{z}}) > = 0$$

all $\alpha,\beta \geq 0$. Here <.,.> is the Hermitian inner product on $\phi^{-1}T^{(1,0)}N$ and ∇ the pull-back connection on $\phi^{-1}T^{(1,0)}N$.

We now introduce some related forms on M:

i) *real case*: Let $\mathcal{Q}^k_\phi = (\nabla^k d\phi, \nabla^k d\phi)^{(2k+2,0)} \in C^\infty(\otimes^{2k+2}T^{(1,0)}M)$ where (.,.) is as above and ∇ is the induced covariant derivative on some $\otimes^{i}T^*M \otimes \phi^{-1}TN$.

ii) *complex case*: if N is Kahler let $\mathcal{P}^k_\phi = (\nabla^k \partial\phi, \overline{\partial}\phi)^{(k+2,0)} \in C^\infty(\otimes^{k+2}T^{(1,0)}M)$ where $\partial\phi, \overline{\partial}\phi, (.,.)$ are as above and each ∇ is the covariant differential on some $\otimes^i T^*M \otimes \phi^{-1}T^{(1,0)}N$.

Remark $\quad \mathcal{Q}^0_\phi = \mathcal{P}^0_\phi = \phi^* h^{(2,0)}$, $\mathcal{Q}^1_\phi$ is the quartic differential considered by Bryant [11] and $\mathcal{P}^1_\phi$ is the (3,0) differential considered by Chern-Wolfson [16].

Proposition 2.3 $\quad$ i) ϕ is real isotropic iff each $\mathcal{Q}^k_\phi$ vanishes identically.

ii) ϕ is complex isotropic iff each $\mathcal{P}^k_\phi$ vanishes identically.

Proof $\quad$ i) An easy induction shows that

$$\nabla^k d\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}) = \nabla^k_{\frac{\partial}{\partial z}} d\phi \frac{\partial}{\partial z} + \sum_{i\le k-1} f_i \nabla^i_{\frac{\partial}{\partial z}} d\phi \frac{\partial}{\partial z}$$

for some functions f_i. Thus if ϕ is real isotropic then all

$$(\nabla^k d\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}),\ \nabla^k d\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}))$$

vanish and so each $\mathcal{Q}^k_\phi$ vanishes identically.

Conversely if each $\mathcal{Q}^k_\phi$ vanishes we will show that

$$\eta_{\alpha,\beta} = (\nabla^\alpha_{\frac{\partial}{\partial z}} d\phi \frac{\partial}{\partial z},\ \nabla^\beta_{\frac{\partial}{\partial z}} d\phi \frac{\partial}{\partial z}) = 0 \quad \text{by induction on } \alpha+\beta.$$

First, $\eta_{0,0} = \mathcal{Q}^0_\phi = 0$. Suppose $\eta_{\alpha,\beta} = 0$, $\alpha+\beta < 2k$, then

$$\eta_{k,k} = (\nabla^k_{\frac{\partial}{\partial z}} d\phi \frac{\partial}{\partial z}, \nabla^k_{\frac{\partial}{\partial z}} d\phi \frac{\partial}{\partial z}) = (\nabla^k d\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}), \nabla^k d\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}))$$

and so $\eta_{k,k} = 0$. Now it is easy to see that $\eta_{\alpha,\beta} = 0$ for all $\alpha+\beta = 2k$ and the induction is complete.

A similar argument establishes (ii). $\square$

Having generalised Proposition 2.1 (i) let us examine the holomorphicity of the $\mathcal{P}^k_\phi$, $\mathcal{Q}^k_\phi$.

Theorem 2.4 * i) Let N have constant sectional curvatures. If $\phi: M \to N$ is harmonic and $\mathcal{Q}^0_\phi = \ldots = \mathcal{Q}^{k-1}_\phi = 0$ then $\mathcal{Q}^k_\phi$ is a holomorphic differential.

ii) Let N be a Kahler manifold of constant holomorphic sectional curvature. If $\phi: M \to N$ is harmonic and $\mathcal{P}^0_\phi = \mathcal{P}^1_\phi = \ldots = \mathcal{P}^{k-1}_\phi = 0$ then $\mathcal{P}^k_\phi$ is a holomorphic differential.

Proof A straightforward induction, shows that, without hypotheses on the curvature of N, for $k \geq 1$, if $\mathcal{Q}^0_\phi = \ldots = \mathcal{Q}^{k-1}_\phi = 0$ then

$$(\nabla^i d\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}),\ \nabla^k d\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z})) = 0 \quad \text{for } i < k. \tag{3}$$

Now
$$\frac{\partial}{\partial \bar z}\mathcal{Q}^k_\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}) = \frac{\partial}{\partial \bar z}\,(\nabla^k d\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}),\ \nabla^k d\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}))$$
$$= 2(\nabla_{\frac{\partial}{\partial \bar z}}\nabla^k d\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}),\ \nabla^k d\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}))\ .$$

We claim that

$$\nabla_{\frac{\partial}{\partial \bar z}}\nabla^k d\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}) = \sum_0^{k-1} f_i \nabla^i d\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}) \text{ for some functions } f_i.$$

This, together with (3), shows that $\frac{\partial}{\partial \bar z}\mathcal{Q}^k_\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}) = 0$ and establishes the theorem.

To prove the claim we induct on k: the case $k = 0$ follows from the harmonicity of ϕ. Now suppose that $\nabla_{\frac{\partial}{\partial \bar z}}\nabla^\rho d\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}) = \sum_0^{\rho-1} f_i \nabla^i d\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z})$ for $\rho < k$. Then

* This theorem was discovered by Wood [75] independently.

$$\nabla_{\frac{\partial}{\partial \bar z}} \nabla^k d\phi\left(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial \bar z}\right) = \nabla_{\frac{\partial}{\partial \bar z}} \nabla_{\frac{\partial}{\partial z}} \nabla^{k-1} d\phi\left(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}\right) - \nabla_{\frac{\partial}{\partial \bar z}} (k-1) f\, \nabla^{k-1} d\phi\left(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}\right)$$

where $f = \langle \nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \rangle$,

$$= \nabla_{\frac{\partial}{\partial z}} \nabla_{\frac{\partial}{\partial \bar z}} \nabla^{k-1} d\phi\left(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}\right) + \phi^* R\left(\frac{\partial}{\partial \bar z},\frac{\partial}{\partial z}\right) \nabla^{k-1} d\phi\left(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}\right)$$

$$- \nabla_{\frac{\partial}{\partial \bar z}} (k-1) f\, \nabla^{k-1} d\phi\left(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}\right)$$

$$= \sum_{i \leq k-1} g_i \nabla^i d\phi\left(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}\right) + \phi^* R\left(\frac{\partial}{\partial \bar z},\frac{\partial}{\partial z}\right) \nabla^{k-1} d\phi\left(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}\right)$$

by the induction hypothesis.

Since N has constant sectional curvatures,

$$\phi^* R\left(\frac{\partial}{\partial \bar z},\frac{\partial}{\partial z}\right) \nabla^{k-1} d\phi\left(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}\right) = c\,\Big\{ \left(d\phi\left(\frac{\partial}{\partial z}\right), \nabla^{k-1} d\phi\left(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}\right)\right) d\phi\left(\frac{\partial}{\partial z}\right)$$

$$- \left(d\phi\left(\frac{\partial}{\partial z}\right), \nabla^{k-1} d\phi\left(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}\right)\right) d\phi\left(\frac{\partial}{\partial \bar z}\right)\Big\}$$

the second summand of which vanishes by (3). Thus the induction is completed and so is part (i) of the theorem.

ii) The proof of (ii) is similar but slightly simpler.

$P^k_\phi\left(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}\right) = \langle \nabla^k \partial\phi\left(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}\right), \partial\phi\left(\frac{\partial}{\partial \bar z}\right)\rangle$ and since ϕ is harmonic

$$\frac{\partial}{\partial \bar z} P_\phi\left(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}\right) = \langle \nabla_{\frac{\partial}{\partial \bar z}} \nabla^k \partial\phi\left(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}\right), \partial\phi\left(\frac{\partial}{\partial \bar z}\right)\rangle$$

Now it suffices to show that if $P_\phi^o = \ldots = P_\phi^{k-1}$ then

$$\nabla_{\frac{\partial}{\partial \bar{z}}} \nabla^k \partial\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}) = \sum_o^{k-1} g_i \nabla^i \partial\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z})$$ to show which we

proceed as above, noting that if N has constant holomorphic sectional curvatures

$$\phi^* R(\frac{\partial}{\partial \bar{z}},\frac{\partial}{\partial z})\nabla^{k-1}\partial\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}) = c\Big\{<\nabla^{k-1}\partial\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}), \frac{\partial\phi}{\partial \bar{z}}>\frac{\partial\phi}{\partial \bar{z}} -$$

$$<\nabla^{k-1}\partial\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z}), \frac{\partial\phi}{\partial z}>\frac{\partial\phi}{\partial z} - k(\phi)\nabla^{k-1}\partial\phi(\frac{\partial}{\partial z},\ldots,\frac{\partial}{\partial z})\Big\}$$

where $k(\phi) = <\frac{\partial\phi}{\partial z}, \frac{\partial\phi}{\partial z}> - <\frac{\partial\phi}{\partial \bar{z}} \frac{\partial\phi}{\partial \bar{z}}>$. □

Corollary 2.5 Let M be a connected Riemann surface and N a real (complex) space form and let $\phi: M \to N$ be harmonic. If ϕ is real (complex) isotropic on a non-empty open set then ϕ is real (complex) isotropic on M.

Proof Combine propositions 2.3, 2.4 and theorem 2.4. □

CHAPTER 5

f-STRUCTURES AND HARMONIC MAPS

Manifolds with f-structures (introduced by Yano [77]) and f-holomorphic maps between them are a natural generalisation of almost Hermitian manifolds and holomorphic maps. In this chapter, which serves as an introduction to Chapters 6 and 7 in which f-holomorphic maps play a leading role, we consider conditions on the f-structure that guarantee the harmonicity of f-holomorphic maps. There are two such conditions (one due to Rawnsley [59]), which are, in some sense complementary, although both are natural generalisations of the Kähler condition for Hermitian manifolds.

1. *f-structures and f-holomorphic maps*

Let (N,h) be a Riemannian manifold.

Definitions An *f-structure* on N is a smooth, constant rank skew-symmetric section of $\mathrm{End}(TN)$, denoted F such that

$$F^3 + F \equiv 0.$$

F has eigenvalues $+i,-i$ and zero and thus the complexified tangent space of N splits as an orthogonal direct sum of eigenspaces:

$$TN^{\mathbb{C}} = TN^+ + TN^- + TN^0 \quad \text{where } TN^+ = \overline{TN^-}.$$

The complex dimension of TN^+ is the *rank* of F.

F is *Cauchy-Riemann (C.R.) integrable* if TN^+ is closed under Lie bracket.

F is *Levi-flat* if $\mathrm{Im}F$ is closed under Lie bracket (that is integrable in the sense of Frobenius).

The *fundamental 2-form of F*, denoted ω, is given by

$$\omega(X,Y) = h(X,FY) \quad \text{for} \quad X,Y \in TN.$$

An *f-Kähler* f-structure is one with closed fundamental 2-form.
Lastly, let ∇ denote the Levi-Civita connection on N. Then F satisfies *condition A* if

$$\nabla_X \, C^\infty(TN^+) \subset C^\infty(TN^+) \quad \text{for all } X \in TN^- .$$

Remarks

(i) Let $\dim N = n$ and F be a rank K f-structure on N.
If $2K=n$, F is an almost Hermitian structure.
If $2K+1=n$, F is a Cauchy-Riemann (C.R.) structure or equivalently an almost contact metric structure (c.f. Blair [9]).

(ii) If F is an almost Hermitian structure, F is clearly Levi-flat and is C.R. integrable if and only if it is an integrable complex structure. Lastly, if ω is the Kähler form of F, condition A (which is due to Rawnsley [59]) reduces to the familiar condition that the (1,2) part of $d\omega$ vanish, i.e. that F be (1,2) symplectic in the sense of Eells-Salamon [26].

(iii) If F is a rank K f-structure with $2K < \dim N$, C.R. integrability is a substantially weaker condition than the vanishing of the Nijenhuis tensor of F.

Examples

(i) Let N be a real hypersurface of an almost Hermitian manifold $(\overline{N},h,J)$ equipped with the induced metric. We define a co-dimension one sub-bundle H of TN by

$$H = TN \cap JTN.$$

Letting π denote orthogonal projection onto H we can define a C.R. structure F on TN by

$$F = J \circ \pi .$$

Then it is easy to see that if $\overline{N}$ is integrable, then F is C.R. integrable and since the fundamental 2-form of F is just the restriction of the Kähler form of $\overline{N}$ to N, we see that if $\overline{N}$ is almost Kähler, N is f-Kähler. In particular, real hypersurfaces of $\mathbb{C}^n$ acquire integrable f-Kähler C.R. structures in this way.

(ii) As noted above, almost contact manifolds are examples of manifolds with f-structures. In particular, the quasi-Sasaki manifolds studied by Blair and Goldberg carry integrable f-Kähler C.R. structures. (See [8] and [10].)

(iii) The constructions of Chapter 6 will provide manifolds with f-structures, sometimes satisfying condition A, which arise as bundles of f-structures over Riemannian manifolds.

Definition Let (M,g) and (N,h) be Riemannian manifolds with f-structures F^M, F^N respectively and let $\phi : M \to N$ be a smooth map. ϕ is *f-holomorphic* if

$$d\phi \circ F^M = F^N \circ d\phi$$

or, equivalently, if

$$\phi_* TM^+ \subset TN^+, \ \phi_* TM^O \subset TN^O, \ \phi_* TM^- \subset TN^- .$$

Remark For F^M, F^N C.R. structures, this notion is stronger than that of a *C.R. map* used by several authors (see eg. [15]), the definition of which requires no conditions on the image of TM^O.

Now let (N,h) be orientable and let F be a C.R. structure on N. Let ξ denote a fixed section of TN^O of unit length. Then we may measure the non-integrability of ImF as follows:

Definition Let η denote the covariant representative of ξ and define the *Levi-form* of F by

$$L^F(X,Y) = d\eta(X,FY) \text{ for } X,Y \in \text{Im}F.$$

N is *pseudo-convex* if L^F is definite on ImF.

It is clear that F is Levi-flat if and only if L^F vanishes identically.*
Further, if F is integrable then it is easy to see that L^F is symmetric and invariant under F. In this case, our definition coincides with that of Chern-Moser [15].

Example Consider $S^{2n-1} \subset \mathbb{C}^n$ with the C.R. structure, F, induced from $\mathbb{C}^n$. If we put

$$\xi = in$$

where n is the outward unit vector field on S^{2n-1} we find that L^F coincides with the metric on S^{2n-1} and thus S^{2n-1} is pseudo-convex.

The behaviour of the Levi-form determines to some extent the rank of f-holomorphic maps into C.R. manifolds. For example, we have

Proposition 1.1 Let (M,g),(N,h) be Riemannian manifolds and suppose M has a Levi-flat f-structure and N a pseudo-convex C.R. structure. Then if $\phi : M \to N$ is f-holomorphic, $d\phi$ vanishes on ImF^M. In particular, if M is almost Hermitian, all f-holomorphic maps $\phi : M \to N$ are constant.

Proof Let $\eta \in C^\infty(T^*N)$ be as above then since η vanishes on ImF^N and ϕ is f-holomorphic we have that $\phi^*\eta$ vanishes on ImF^M. Denoting by L^N the Levi-form on N, we have, for $X,Y \in \text{ImF}^M$:

* since $L^F(X,Y) = h([X,FY],\xi)$

$$\begin{aligned}\phi^*L(X,Y) &= d\eta(\phi_*(X), F^N\phi_*(Y)) \\ &= d\eta(\phi_*(X), \phi_*(F^M Y)) \\ &= \phi^* d\eta(X, F^M Y) \\ &= d\phi^*\eta(X, F^M Y) = 0,\end{aligned}$$

since $\mathrm{Im}F^M$ is closed under Lie bracket. Thus $\phi_*\mathrm{Im}F^M$ is contained in the isotropy subspace of L^N and therefore vanishes. □

Example There are no non-constant f-holomorphic maps from a Riemann surface into S^{2n-1}.

2. *Condition A and f-holomorphic harmonic maps*

Rawnsley has proved the following theorem:

Theorem 2.1 [59] Let (M,g) be an almost Hermitian manifold with co-closed Kähler form and (N,h) a Riemannian manifold with f-structure satisfying condition A. Then any f-holomorphic map from $M \to N$ is harmonic.

However, condition A is quite restrictive as the following proposition shows:

Proposition 2.2 An integrable f-structure satisfying condition A is Levi-flat.

Proof Let F be such an f-structure on a manifold (N,h) and let $X,Y \in C^\infty(TN^+)$. Since F is integrable we have

$$[X,Y] \in C^\infty(TN^+)$$

and by condition A we have

$$[X,\bar{Y}] = \nabla_X\bar{Y} - \nabla_{\bar{Y}}X \in C^\infty(TN^+ + TN^-)$$

so that $C^\infty(TN^+ + TN^-)$ is closed under Lie bracket whence ImF is also. □

Corollary 2.3 Let $\dim N=3$, then any f-structure on N satisfying condition A is Levi-flat.

Proof Any f-structure is necessarily integrable since $\dim TN^+$ is one. □

Corollary 2.4 The natural integrable f-Kähler C.R. structure on S^{2n-1} does not satisfy condition A.

Proof Immediate from the pseudo-convexity of S^{2n-1}. □

Thus we now turn to f-holomorphic maps into f-Kähler manifolds.

3. *f-Kahler manifolds and a homotopy invariant*

Let (N,h) be a Riemannian manifold with f-structure F and (M,g,J) be an almost Hermitian manifold, and let $\phi : M \to N$ be a smooth map. Corresponding to the decomposition of $TN^{\mathbb{C}}$ into the eigenspaces of F, we may decompose ϕ_* into components

$$\phi_*^+ : TM^{\mathbb{C}} \to TN^+$$
$$\phi_*^- : TM^{\mathbb{C}} \to TN^-$$
$$\phi_*^o : TM^{\mathbb{C}} \to TN^o$$

by composing ϕ_* with the appropriate projections.

Now let $E_1,\ldots,E_n$, $JE_1,\ldots,JE_n$ be an orthonormal frame field for M and let

$$Z_i = E_i - i\,JE_i.$$

Then the energy density of $\phi, e(\phi)$, is given by

$$
\begin{aligned}
e(\phi) &= \tfrac{1}{2}\text{Trace } \phi^{*}h \\
&= \tfrac{1}{2}h(\phi_{*}(Z_i), \phi_{*}(\overline{Z}_i)) \\
&= \tfrac{1}{2}\, h\,(\phi^{o}_{*}(Z_i), \phi^{o}_{*}(\overline{Z}_i)) + \tfrac{1}{2}h(\phi^{+}_{*}(Z_i), \phi^{-}_{*}(\overline{Z}_i)) + \tfrac{1}{2}h(\phi^{-}_{*}(Z_i), \phi^{+}_{*}(\overline{Z}_i)).
\end{aligned}
$$

The three summands of this last equation are denoted by $e^{o}(\phi), e^{+}(\phi), e^{-}(\phi)$ respectively and since $Z_1, \ldots, Z_n$ span $T^{(1,0)}M$, it is clear that ϕ is f-holomorphic if and only if $e^{o}(\phi)$ and $e^{-}(\phi)$ vanish identically and ϕ is anti-f-holomorphic if and only if $e^{o}(\phi)$ and $e^{+}(\phi)$ vanish identically. (ϕ is anti-f-holomorphic if and only if ϕ is f-holomorphic with respect to the conjugate complex structure on M.)

Lemma 3.1 Let ω^{M}, ω^{N} denote the fundamental two forms of M and N respectively. Then

$$
\langle \omega^{M}, \phi^{*}\omega^{N} \rangle = e^{+}(\phi) - e^{-}(\phi).
$$

Proof With $E_1, \ldots, E_n$, $JE_1, \ldots, JE_n$ a frame field for M as above we have

$$
\begin{aligned}
\langle \omega^{M}, \phi^{*}\omega^{N} \rangle &= -\phi^{*}\omega^{N}(E_i, JE_i) \\
&= -\frac{1}{2i}\,\phi^{*}\omega^{N}(Z_i, \overline{Z}_i) \\
&= -\frac{1}{2i}\, h(\phi_{*}(Z_i),\ F\phi_{*}(\overline{Z}_i)) \\
&= -\frac{1}{2i}\,[-ih(\phi^{+}_{*}(Z_i), \phi^{-}_{*}(\overline{Z}_i)) + ih(\phi^{-}_{*}(Z_i), \phi^{+}_{*}(\overline{Z}_i))] \\
&= e^{+}(\phi) - e^{-}(\phi).
\end{aligned}
$$

□

Now put

$$E^{+}(\phi) = \int_M e^{+}(\phi) * 1 \ , \ E^{-}(\phi) = \int_M e^{-}(\phi) * 1 \ , \ E^{o}(\phi) = \int_M e^{o}(\phi) * 1$$

and

$$K(\phi) = E^{+}(\phi) - E^{-}(\phi).$$

Following Lichnerowicz [50], we have

Theorem 3.2 Let (M,g) be almost Hermitian with co-closed fundamental 2-form and let (N,h) be f-Kähler. Then $K(\phi)$ only depends on the homotopy class of ϕ.

Proof By Lemma 3.1, we have

$$K(\phi) = \int_M <\omega^M, \phi^* \omega^N> * 1.$$

Now, let ϕ_t be a smooth variation of ϕ. Then

$$\frac{\partial}{\partial t} K(\phi_t) = \frac{\partial}{\partial t} \int_M <\omega^M, \phi_t^* \omega^N> * 1$$

$$= \int_M <\omega^M, \frac{\partial}{\partial t} \phi_t^* \omega^N> * 1$$

and since ω^N is closed we have, by the homotopy lemma of Lichnerowicz [50] that

$$\frac{\partial}{\partial t} \phi_t^* \omega^N = d(\phi_t^* i(\frac{\partial \phi}{\partial t}) \omega^N)$$

and so

$$\frac{\partial}{\partial t} K(\phi_t) = \int_M <d^* \omega^M, \phi_t^* i(\frac{\partial \phi}{\partial t}) \omega^N> * 1 = 0$$

since ω^M is co-closed. Thus $K(\phi_t)$ is constant and the theorem follows.

□

Corollary 3.3 Let (M,g) be almost Hermitian with co-closed Kähler form and (N,h) be f-Kähler. If $\phi : M \to N$ is a smooth map then we have

(i) If ϕ is f-holomorphic or anti-f-holomorphic, ϕ is harmonic and minimises energy in its homotopy class.

(ii) If ϕ minimises energy in its homotopy class and is homotopic to an (anti)f-holomorphic map then ϕ is (anti)f-holomorphic.

(iii) Let ϕ be (anti)f-holomorphic and ϕ_t a smooth variation of ϕ through harmonic maps, then each ϕ_t is (anti)f-holomorphic.

(iv) Let ϕ_1 be f-holomorphic and ϕ_2 be (anti)f-holomorphic, then if ϕ_1 and ϕ_2 are homotopic, they are both constant.

(v) Let K vanish on some homotopy class $\mathcal{H}$. Then any (anti) f-holomorphic map in $\mathcal{H}$ is constant. In particular, any homotopically trivial (anti)f-holomorphic map is constant.

Proof

$$\begin{aligned} E(\phi) &= E^+(\phi) + E^-(\phi) + E^o(\phi) \\ &= 2E^+(\phi) - K(\phi) + E^o(\phi) \\ &= 2E^-(\phi) + K(\phi) + E^o(\phi). \end{aligned}$$

Thus E, $2E^+ + E^o$, $2E^- + E^o$ differ by constants on each homotopy class and so have the same minima and the same critical points in each homotopy class.

Now since E^+, E^-, E^o are all non-negative and ϕ is f-holomorphic if and only if $2E^- + E^o$ vanishes, we have that such a ϕ is an absolute minimum for $2E^- + E^o$ and thus for E in its homotopy class. Thus (i) is proved.

For (ii), if ϕ minimises E and is homotopic to an f-holomorphic map then $2E^- + E^o$ has minimum value zero which ϕ must attain whence ϕ is f-holomorphic.

For any smooth variation ϕ_t where each ϕ_t is harmonic, $E(\phi_t)$ is constant and thus so are $2E^+(\phi_t) + E^o(\phi_t)$ and $2E^-(\phi_t) + E^o(\phi_t)$, so if ϕ_o is (anti) f-holomorphic each ϕ_t is (anti) f-holomorphic also and (iii) follows.

For (v), if ϕ is (anti) f-holomorphic, then

$$E(\phi) = \pm K(\phi)$$

and so if $K(\phi)$ vanishes, ϕ is constant.

Lastly if ϕ_1, ϕ_2 are homotopic with ϕ_1 f-holomorphic and ϕ_2 (anti) f-holomorphic, then

$$E^+(\phi_1) = K(\phi_2) = K(\phi_2) = -E^-(\phi_2),$$

whence $K(\phi_1)$ vanishes and (iv) follows from (v). □

Now if $\phi^* \omega^N$ is exact, we have

$$K(\phi) = \int_M \langle \omega^M, \phi^* \omega^N \rangle * 1 = \int_M \langle \omega^M, d\alpha \rangle * 1 = \int_M \langle d^* \omega^M, \alpha \rangle * 1 = 0.$$

In particular we have

Corollary 3.4 Let (M,g) be almost Hermitian with co-closed Kähler form and (N,h) an f-Kähler manifold with $H^2(N, \mathbb{R}) = 0$. Then any (anti) f-holomorphic map $\phi : M \to N$ is constant.

Thus taking $N = S^{2n-1}$, we have a second proof of proposition 1.1 in this case, which is valid even if we deform S^{2n-1} into an ellipsoid.

Remark In view of the above results, it is reasonable to ask whether any f-holomorphic maps from Hermitian manifolds into f-Kähler manifolds exist. Using the methods of Chapters 6 and 7, it is possible to show that the Grassman bundle of 2-planes over S^n admits an integrable f-Kähler f-structure and the Gauss maps of totally umbilic conformal immersions of surfaces into S^n provide examples of f-holomorphic maps.

CHAPTER 6

TWISTOR BUNDLES AND GENERALISED GAUSS MAPS

The classical connection between the theory of minimal surfaces in $\mathbb{R}^n$ and complex function theory is well-known. In recent years, this approach has led to Calabi's classification of isotropic minimal immersions into spheres [13] and the generalisation of Calabi's techniques to the classification of isotropic harmonic maps into complex projective spaces and complex Grassmanians by Eells-Wood [29] and Erdem-Wood [31] respectively.

Similarly, self-dual Yang-Mills fields on S^4 can be classified by certain holomorphic vector bundles on $\mathbb{C}P^3$ [6].

All these constructions may be interpreted as an association of differential-geometric data on a manifold N (eg. harmonic maps into N) with holomorphic data (eg. holomorphic maps) on a bundle of complex structures over N, a twistor bundle.

More recently Eells-Salamon [25] and J.H. Rawnsley [59] have greatly extended the possibilities of this theory by considering non-integrable complex structures on twistor bundles and constructing twistor bundles of f-structures. In this and the next chapter we will apply these techniques to classify certain conformal harmonic maps and to construct such maps. Throughout we will adopt the approach of Rawnsley [59]. The proofs of most assertions concerning the construction and properties of the twistor bundles under consideration will be given in the Appendix to this chapter.

1. *The bundle of f-structures*

Let V be an n-dimensional euclidean space. Let O(V) be the orthogonal group of V with Lie algebra o(V) of skew-symmetric linear transformations.

Let $F_k(V)$ denote

$$\{F \in o(V) : F^3 + F = 0 \text{ and } \operatorname{rank} F = 2k\},$$

the space of *rank k f-structures*. Observe that if $2k = n$ $F_k(V)$ is the space of Hermitian almost complex structures on V and if $2k+1 = n$, $F_k(V)$ is the space of C.R. structures compatible with the inner product on V (almost contact metric structures on V).

If $F \in F_k(V)$, F has eigenvalues $+i, -i, 0$ and $V^{\mathbb{C}}$ splits as an orthogonal direct sum of eigenspaces:

$$V^{\mathbb{C}} = V_F^+ + V_F^- + V_F^0 \quad \text{with} \quad \text{(i)} \ V_F^- = \overline{V_F^+} \ \text{and}$$

$$\text{(ii)} \ \dim_{\mathbb{C}} V_F^+ = k.$$

It follows immediately from the orthogonality and (i) that $(V_F^+, V_F^+) = 0$ where $(\ ,\ )$ is the complex bilinear extension of the inner product on V, that is V_F^+ is *isotropic*.

In fact, the correspondence $F \to V_F^+$ is a bijection between $F_k(V)$ and the space of k-dimensional isotropic subspaces of $V^{\mathbb{C}}$.

O(V) acts transitively on $F_k(V)$ by conjugation. Fixing $F_o \in F_k(V)$ and denoting by H_{F_o} the stabiliser of F_o in O(V) we see that $F_k(V)$ is a homogeneous space $\frac{O(V)}{H_{F_o}}$. Note that $H_{F_o} \cong U(k) \times O(n-2k)$. In fact, since $F_k(V)$ is an adjoint orbit of a compact Lie group in its Lie algebra, $F_k(V)$ is a reductive homogeneous space with an O(V) invariant complex structure and invariant Kähler metric (see Appendix).

Remark If $2k < n$, $SO(V)$ is transitive on $F_k(N)$ and we can identify $F_k(V)$ with $\frac{SO(n)}{SO(n-2k) \times U(k)}$. However $\frac{SO(2n)}{U(n)}$ is the space of hermitian almost complex structures with fixed natural orientation.

Now let (N,h) be a Riemannian manifold with orthonormal frame bundle $O(N,h)$. Let $\pi : F_k(N,h) \to N$ be the bundle whose fibre at $x \in N$ is $F_k(T_xN)$: the bundle of rank k f-structures on N.

Then $F_k(N,h) = O(N) \times_{O(n)} \frac{O(n)}{H_{F_o}}$, for some fixed $F_o \in F_k(\mathbb{R}^n)$.

Remark Since $SO(n)$ is transitive on $F_1(\mathbb{R}^n)$ it is easy to see that $F_1(N,h)$ is isomorphic to the Grassmann bundle $G_2(TN)$ of oriented real 2-planes in TN. The f-structure corresponding to a given plane is defined by projection onto the plane followed by a rotation of $\frac{\pi}{2}$ in a positive sense.

The Levi-Civita connection on N, ∇^N, induces an $O(n)$ invariant horizontal distribution on $O(N)$ and thus a horizontal distribution on all associated bundles. In particular, we have a splitting of $TF_k(N,h)$ into a direct sum of sub-bundles:

$$TF_k(N,h) = \mathcal{V} + \mathcal{H}$$

where $\mathcal{V}$ is the tangent distribution to the fibres.

Let $E \to F_k(N,h)$ denote the pullback of the tangent bundle of N by π. Then E is equipped with a tautological f-structure $\mathcal{F}$ given by

$$\mathcal{F} \mid E_F = F$$

where we identify E_F with $T_{\pi(F)}N$.

The vertical distribution V inherits a complex structure from the $O(n)$ invariant complex structure on $F_k(\mathbb{R}^n)$ denoted by J.
Thus we define two f-structures on $F_k(N,h)$ given by

$$F_1 = \begin{cases} J & \text{on } V \\ F & \text{on } H \end{cases}$$

$$F_2 = \begin{cases} -J & \text{on } V \\ F & \text{on } H \end{cases},$$

where we identify H with E via $d\pi$.

The metric on $F_k(N,h)$ is any metric for which V and H are orthogonal, J is Hermitian and π is a Riemannian fibration.

Now $O(N) \times_{O(n)} \dfrac{O(n)}{H_{F_o}}$ can be identified with $\dfrac{O(N)}{H_{F_o}}$ and thus $O(N)$ is a principal H_{F_o} bundle over $F_k(N,h)$. Composing the Levi-Civita connection form on $O(N)$ with projection on $h_{F_o} \subset o(n)$, where h_{F_o} is the Lie algebra of H_{F_o}, gives an H_{F_o}-connection on $F_k(N,h)$ denoted by D.

Viewing V as a sub-bundle of $\mathrm{End}(E)$ and denoting by P, projection of $TF_k(N,h)$ onto V we have

Proposition 1.1 [59] On E, $D = \pi^{-1}\nabla^N - P$. Further, D preserves horizontal and vertical distributions and commutes with F_1, F_2.

Now suppose that (N^{2n},h) is a Kähler manifold and let $\pi_1 : \mathbb{G}_r(T^{(1,0)}N) \to N$ be the Grassmann bundle of complex r-planes in $T^{(1,0)}N$. We define a fibre map $j : \mathbb{G}_r(T^{(1,0)}N) \to J(N) = F_n(N^{2n},h)$ as follows. For an r-plane $W \subset T_x^{(1,0)}N$

$$j(W) = \begin{cases} i & \text{on } W + \overline{(W^{\perp} \cap T^{(1,0)}N)} \\ -i & \text{on } \overline{W} + (W^{\perp} \cap T^{(1,0)}N). \end{cases}$$

Thus $j(W)$ is an almost complex structure on T_xN which commutes with the ambient complex structure.

Theorem 1.2 [54] The image of $\mathbb{G}_r(T^{(1,0)}N)$ is invariant by both complex structures J_1, J_2 on $J(N)$.

Proof *See Appendix.*

2. *Twistor Bundles over Grassmannians*

A. *Real Grassmannians*

Let $G_r(\mathbb{R}^n)$ denote the set of oriented r-dimensional subspaces of $\mathbb{R}^n$. $SO(n)$ acts transitively on $G_r(\mathbb{R}^n)$ and $G_r(\mathbb{R}^n)$ becomes a Riemannian symmetric space: $\dfrac{SO(n)}{SO(r) \times SO(n-r)}$. Note that $G_1(\mathbb{R}^n)$ is isometric to S^{n-1}.

Let $T \to G_r(\mathbb{R}^n)$ denote the tautological sub-bundle of $G_r(\mathbb{R}^n) \times \mathbb{R}^n$ whose fibre at $W \in G_r(\mathbb{R}^n)$ is $W \subset \mathbb{R}^n$. As is well-known, $TG_r(\mathbb{R}^n)$ is isomorphic to $T \otimes T^\perp \cong L(T,T^\perp)$. Now let $\pi_1 : F_k(T^\perp) \to G_r(\mathbb{R}^n)$ denote the bundle whose fibre at W is $F_k(W^\perp)$. We have

Theorem 2.1 There exists a horizontal distribution H on $F_k(T^\perp)$ and f-structures F_1^r, F_2^r with $F_1^r = F_2^r$ on H, $F_1^r = -F_2^r$ on V, the vertical distribution, such that

(i) F_1^r is integrable and if $2k = n-r$, F_1^r is a Kählerian complex structure,

(ii) F_2^r satisfies condition A,

(iii) The map $j : F_k(T^\perp) \to F_{rk}(G_r(\mathbb{R}^n))$ given by

$$j(F) = \mathrm{id} \otimes F \text{ on } W \otimes W^\perp \text{ where } W = \pi(F),$$

is f-holomorphic with respect to (F_1^r, F_1) and (F_2^r, F_2).

This is a special case of a general construction for symmetric spaces considered in the Appendix (A3,A4) wherein a proof of theorem 2.1 is provided. For the moment we shall just describe the f-structures:

We have sub-bundles of $F_k(T^\perp) \times \mathbb{R}^n$ given by $\pi_1^{-1}T$ and $\pi_1^{-1}T^\perp = E$.

Further, $E^{\mathbb{C}}$ splits under the obvious tautological f-structure into eigenbundles E^+, E^-, E^0. Then it can be shown that $TF_K(T^\perp)^{\mathbb{C}}$ is naturally isomorphic to

$$\pi_1^{-1}T \otimes E^{\mathbb{C}} + (\Lambda^2E^+ + \Lambda^2E^- + E^+ \otimes E^0 + E^- \otimes E^0)$$

where the terms in the bracket correspond to $V^{\mathbb{C}}$ and $\pi_1^{-1}T \otimes E^{\mathbb{C}}$ to $H^{\mathbb{C}}$.

$$\text{Put } \quad F_1^r = F_2^r = i \quad \text{on} \quad \pi_1^{-1}T \otimes E^+$$
$$F_1^r = -F_2^r = i \text{ on } \Lambda^2E^+ + E^+ \otimes E^0.$$

Now fix r,s,k such that $r+s+2k=n$ and let $\pi_r:T_r \to G_r(\mathbb{R}^n)$, $\pi_s:T_s \to G_s(\mathbb{R}^n)$ be the corresponding tautological bundles. We may identify $F_k(T_r^\perp)$ and $F_k(T_s^\perp)$ as follows: for $F \in F_k(W^\perp)$ let $V = \ker F$ and define $\tilde{F} \in F_k(V^\perp)$ by

$$\tilde{F} = F \text{ on } \operatorname{Im}F$$
$$\tilde{F} = 0 \text{ on } V.$$

Here we orient V so that it is compatible with the natural orientations on $\operatorname{Im}F, W$ and $\mathbb{R}^n$.

Identifying $F_k(T_s^{\perp})$ with $F_k(T_r^{\perp})$ in this way and examining the eigenbundles of the f-structures we observe the following.

Proposition 2.2 (i) $F_1^r = F_1^s$

(ii) if $k=1$, $F_2^r = -F_2^s$.

Proof At $F \in F_k(W^\perp) \subset F_k(T_r^{\perp})$,

$$F_1^r = i \quad \text{on} \quad W \otimes E_F^+ + \Lambda^2E_F^+ + E_F^+ \otimes V$$

while at $\tilde{F} \in F_k(V^{\perp}) \subset F_k(T_s^{\perp})$,

$$\mathcal{F}_1^s = i \text{ on } V \otimes E_{\tilde{F}}^+ + \Lambda^2 E_{\tilde{F}}^+ + E_{\tilde{F}}^+ \otimes W .$$

Now $E_{\tilde{F}}^+ = E_F^+$ and thus $\mathcal{F}_1^r = \mathcal{F}_1^s$. The second assertion follows similarly observing that if $k=1$, $\Lambda^2 E^+ = 0$. □

Remark The natural $SO(n)$ action on $F_k(T_s^{\perp})$ induces an isomorphism of $F_k(T_s^{\perp})$ with $\dfrac{SO(n)}{SO(s)\times U(k)\times SO(r)}$, similarly for $F_k(T_r^{\perp})$. Treating

$\dfrac{SO(n)}{SO(s)\times U(k)\times SO(r)}$ as the common twistor space, the twistor fibrations are simply the homogeneous fibrations induced by the inclusions $SO(s)\times U(k) \hookrightarrow SO(n-r)$ and $U(k)\times SO(r) \to SO(n-s)$.

Remark $F_k(T_1^{\perp})$ is clearly identical with $F_k(S^{n-1})$ and thus Theorem 2.1 provides a proof of the (well-known) properties of this bundle.

B. *Complex Grassmannians*

We now repeat the argument with $\mathbb{G}_r(\mathbb{C}^n)$, the set of r-dimensional complex subspaces of $\mathbb{C}^n$. The transitive action of $U(n)$ endows $\mathbb{G}_r(\mathbb{C}^n)$ with the structure of a Hermitian symmetric space $\dfrac{U(n)}{U(r)\times U(n-r)}$. $\mathbb{G}_1(\mathbb{C}^n)$ is isometric to $\mathbb{C}P^{n-1}$.

Let $T_r \to \mathbb{G}_r(\mathbb{C}^n)$ denote as before the tautological sub-bundle of $\mathbb{G}_r(\mathbb{C}^n)\times\mathbb{C}^n$. Then $T^{(1,0)}\mathbb{G}_r(\mathbb{C}^n)$ is isomorphic to $\overline{T}_r \otimes T_r^{\perp} \cong L(T_r, T_r^{\perp})$.

Let $\pi_r : \mathbb{G}_s(T_r^{\perp}) \to \mathbb{G}_r(\mathbb{C}^n)$ denote the complex Grassmann bundle over $\mathbb{G}_r(\mathbb{C}^n)$ whose fibre at W is $\mathbb{G}_s(W^{\perp})$. Then we have

Theorem 2.3 There exists a horizontal distribution $\mathcal{H}$ on $\mathbb{G}_s(T_r^{\perp}) \to \mathbb{G}_r(\mathbb{C}^n)$ and almost complex structures J_1^r, J_2^r on $\mathbb{G}_s(T_r^{\perp})$ with $J_1^r = J_2^r$ on $\mathcal{H}$, $J_1^r = -J_2^r$ on the vertical distribution such that

(i) J_1^r is integrable and Kählerian,

(ii) J_2^r is not integrable but satisfies condition 'A' (ie $d\omega^{1,2} \equiv 0$ for ω the Kähler form with respect to a suitable metric),

(iii) the map $j : \mathbb{G}_s(T_r^{\perp}) \to J(\mathbb{G}_r(\mathbb{C}^n))$ given by

$$j(V) = i \text{ on } \overline{W} \otimes V + W \otimes \overline{(V^{\perp} \cap W^{\perp})}$$

for $V \in G_s(W^{\perp})$ and $W \in G_r(\mathbb{C}^n)$, is holomorphic with respect to (J_1^r, J_1), (J_2^r, J_2).

The reader is referred to the Appendix for the proof of this theorem. The almost complex structures will be described below.

Remark It is easily seen that j factors through $\mathbb{G}_{rs}(T^{(1,0)}\mathbb{G}_r(\mathbb{C}^n))$ by $V \mapsto \overline{W} \otimes V$ followed by the map $\mathbb{G}_{rs}(T^{(1,0)}\mathbb{G}_r(\mathbb{C}^n)) \to J(\mathbb{G}_r(\mathbb{C}^n))$ described in Section 1. Further, if $r = 1$ this provides an isomorphism of $\mathbb{G}_s(T\mathbb{C}P^{(1,0)n-1})$ with $\mathbb{G}_s(T^{\perp})$ which is a biholomorphism for both sets of complex structures.

Let $F(r,s,n-r-s,\mathbb{C}^n)$ denote the flag manifold of ordered triples of mutually orthogonal subspaces of $\mathbb{C}^n$ of dimensions $r,s,n-r-s$ and let $\pi_i : F(r,s,n-r-s,\mathbb{C}^n) \to \mathbb{G}_i(\mathbb{C}^n)$ denote the obvious projections for $i = r,s$ or $n-r-s$.

$\pi_r : \mathbb{G}_s(T_r^{\perp}) \to \mathbb{G}_r(\mathbb{C}^n)$ is naturally isomorphic to $\pi_r : F(r,s,n-r-s,\mathbb{C}^n) \to G_r(\mathbb{C}^n)$ identifying $V \in \mathbb{G}_s(W\ )$, $W \in \mathbb{G}_r(\mathbb{C}^n)$ with $(W,V,(V+W)^{\perp})$ and so we will confuse the two fibrations and define J_1^r, J_2^r on $F(r,s,n-r-s,\mathbb{C}^n)$.

There are tautological bundles $T_r, T_s, T_{n-r-s} = (T_r \oplus T_s)^{\perp}$ on $F(r,s,n-r-s)$, defined in the obvious way, and $TF(r,s,n-r-s)$ is isomorphic to the direct sum of $\overline{T}_r \otimes T_s$, $\overline{T}_s \otimes T_{n-r-s}$, $\overline{T}_r \otimes T_{n-r-s}$ and their complex conjugates. The kernel of $d\pi_r$ is $\overline{T}_s \otimes T_{n-r-s} + T_s \otimes \overline{T_{n-r-s}}$ and the remaining summands form the horizontal distribution.

Now define J_1^r, J_2^r by

$$J_1^r = J_2^r = i \quad \text{on} \quad \overline{T}_r \otimes T_s + T_r \otimes \overline{T_{n-r-s}},$$

$$J_1^r = -J_2^r = i \quad \text{on} \quad T_s \otimes \overline{T_{n-r-s}}.$$

Remark J_1^r is the Kahler structure induced by the inclusion of $F(r,s,n-r-s,\mathbb{C}^n)$ into $\mathbb{G}_{n-r-s}(\mathbb{C}^n) \times \mathbb{G}_{n-s}(\mathbb{C}^n)$ given by

$$(W,V,(W+V)^{\perp}) \mapsto ((W+V)^{\perp}, V^{\perp}).$$

Confusing $F(r,s,n-r-s;\mathbb{C}^n)$ with $F(n-r-s,r,s;\mathbb{C}^n)$ and $F(s,n-r-s,r;\mathbb{C}^n)$ can identify $\mathbb{G}_s(T_r^{\perp})$ with $\mathbb{G}_r(T_{n-r-s}^{\perp})$ and $\mathbb{G}_{n-r-s}(T_s^{\perp})$.

Proposition 2.4 Under these identifications $J_2^r = J_2^s = J_2^{n-r-s}$.

Proof By inspection of the +i eigenbundles: they are

$$J_2^r : \overline{T}_r \otimes T_s + T_r \otimes \overline{T}_{n-r-s} + \overline{T}_s \otimes T_{n-r-s}$$

$$J_2^s : \overline{T}_s \otimes T_{n-r-s} + T_s \otimes \overline{T}_r + \overline{T}_{n-r-s} \otimes T_r$$

$$J_2^{n-r-s} : \overline{T}_{n-r-s} \otimes T_r + \overline{T}_{n-r-s} \otimes \overline{T}_s + \overline{T}_r \otimes T_s\ ,$$

where in each case the last summand is vertical with respect to the given fibration. □

Remark In fact we can say more: giving $\mathbb{G}_r(\mathbb{C}^n)$, $\mathbb{G}_s(\mathbb{C}^n)$ and $\mathbb{G}_{n-r-s}(\mathbb{C}^n)$ their standard complex structures then π_s is J_1^r antiholomorphic and π_{n-r-s} is J_1^r holomorphic and so on.

Remark U(n) acts on the various $\mathbb{G}_s(T_r^{\perp})$ to give a natural isomorphism with the homogeneous space $\frac{U(n)}{U(r)\times U(s)\times U(n-r-s)}$ under which the twistor fibrations correspond to the homogeneous fibrations induced by the inclusions $U(r) \times U(s) \hookrightarrow U(r+s)$ etc.

3. *Maps into twistor bundles*

Let (M,g) be a Riemannian manifold and let $\psi : M \to F_k(N,h)$ be a smooth map, and $\phi : M \to N$ be $\pi \circ \psi$.

Then ψ induces an f-structure on $\phi^{-1}TN$, denoted F_ψ, which is the pull-back of $\mathcal{F}$ on E since $\phi^{-1}TN = \psi^{-1}E$. Thus if E^+, E^-, E^0 denote the bundles of $+i, -i, 0$ eigenspaces of $E^{\mathbb{C}}$, the eigenspaces of F_ψ are $\psi^{-1}E^+, \psi^{-1}E^-, \psi^{-1}E^0$.

Theorem 3.1 [59] Let (M,g) be an almost Hermitian manifold and $\psi : M \to F_k(N,h)$ a smooth map.

(i) ψ is f-holomorphic with respect to $\mathcal{F}_1$ if and only if $\phi_*(Z) \in \psi^{-1}E^+$ for all $Z \in T^{(1,0)}M$ and

$$\phi^{-1}\nabla_Z(C^\infty(\psi^{-1}E^+)) \subset C^\infty(\psi^{-1}E^+) \quad \text{for all } Z \in T^{(1,0)}M.$$

(ii) ψ is f-holomorphic with respect to $\mathcal{F}_2$ if and only if $\phi_*(Z) \in \psi^{-1}E^+$ for all $Z \in T^{(1,0)}M$

$$\phi^{-1}\nabla_{\bar{Z}}(C^\infty(\psi^{-1}E^+)) \subset C^\infty(\psi^{-1}E^+) \quad \text{for all } \forall Z \in T^{(1,0)}M.$$

In particular, ψ is f-holomorphic and horizontal if and only if $\phi_*(Z) \subset \psi^{-1}E^+$ for all $Z \in T^{(1,0)}M$ and ∇ preserves the eigenspaces of F_ψ i.e. $\nabla F_\psi = 0$.

Theorem 3.2 [59] Let (M,g) be an almost Hermitian cosymplectic manifold. Then if $\psi : M \to F_k(N,h)$ is f-holomorphic with respect to $\mathcal{F}_2$, $\phi = \pi \circ \psi$ is harmonic and $\phi^*h^{(2,0)}$ vanishes identically.

These theorems together with the holomorphicity of the various inclusions j enable us to charactarise the f-holomorphicity of maps into the bundles $F_k(T^{\perp})$ and $\mathbb{G}_s(T_r^{\perp})$ discussed in Section 2.

So let $T_r \to \mathbb{G}_r(\mathbb{C}^n)$ be the tautological bundle of $\mathbb{G}_r(\mathbb{C}^n) \times \mathbb{C}^n$. Under the isomorphism of $T^{(1,0)}\mathbb{G}_r(\mathbb{C}^n)$ with $L(T_r, T_r^{\perp})$, the Levi-Civita connection on $\mathbb{G}_r(\mathbb{C}^n)$ corresponds to the connection on $L(T_r, T_r^{\perp})$ induced by the flat connection on $\mathbb{G}_r(\mathbb{C}^n) \times \mathbb{C}^n$.

Now let $\psi : M \to \mathbb{G}_s(T_r^{\perp})$ be a smooth map with $\phi = \pi_r \circ \psi$. We identify ψ with the sub-bundle $\tilde{\psi}$ of $\phi^{-1}T_r^{\perp}$ it induces, that is

$$\tilde{\psi}_x = \psi(x) \subset T^{\perp}_{\phi(x)}.$$

Let $\partial\phi \in C^{\infty}(TM^* \otimes \phi^{-1}L(T_r, T_r^{\perp}))$ denote the projection of $d\phi$ onto $T^{(1,0)}\mathbb{G}_r(\mathbb{C}^n)$. Then we have

Theorem 3.3 Let (M,g) be an almost Hermitian manifold and $\psi : M \to \mathbb{G}_s(T_r^{\perp})$ a smooth map with $\pi_r \circ \psi = \phi$. Then ψ is holomorphic with respect to J_2^r if and only if

$$\mathrm{Im}\ \partial\phi(Z) \subset \tilde{\psi},\ \mathrm{Im}\ \partial\phi(\bar{Z}) \subset \tilde{\psi}^{\perp} \cap \phi^{-1}T_r^{\perp}, \quad \text{for all } Z \in T^{(1,0)}M \quad *$$

and

$$\phi^{-1}\nabla_{\bar{Z}} C^{\infty}(\tilde{\psi}) \subset C^{\infty}(\tilde{\psi}) \quad \text{for all } Z \in T^{(1,0)}M,$$

where ∇ is the natural connection on $T_r^{\perp}$.

Further, ψ is horizontal and holomorphic if and only if, additionally,

$$\phi^{-1}\nabla_X C^{\infty}(\tilde{\psi}) \subset C^{\infty}(\tilde{\psi}) \quad \text{for all } X \in TM$$

* where $\mathrm{Im}\,\partial\phi(x)$ denotes the image of $\partial\phi(x)$ thought of as an element of $L(T_r, T_r^{\perp})$ see p74.

Proof By theorem 2.3, ψ is J_2^r holomorphic if and only if $j \circ \psi : M \to J(\mathbb{G}_r(\mathbb{C}^n))$ is J_2 holomorphic. Now

$$(j \circ \psi)^{-1}E^+ = \phi^{-1}\overline{T}_r \otimes \widetilde{\psi} + \phi^{-1}T_r \otimes \overline{(\widetilde{\psi}^\perp \cap \phi^{-1}T_r{}^\perp)},$$

whence $\phi_*(Z) \in (j \circ \psi)^{-1}E^+$ if and only if

$$\partial\phi(Z) \in \phi^{-1}\overline{T}_r \otimes \widetilde{\psi} = L(\phi^{-1}T_r, \widetilde{\psi})$$

and

$$\partial\phi(\overline{Z}) \in \phi^{-1}\overline{T}_r \otimes (\widetilde{\psi}^\perp \cap \phi^{-1}T_r{}^\perp) = L(\phi^{-1}T_r, \widetilde{\psi}^\perp \cap \phi^{-1}T_r{}^\perp) \; ,$$

or equivalently,

$$\mathrm{Im}\ \partial\phi(Z) \subset \widetilde{\psi} \quad . \quad \mathrm{Im}\ \partial\phi(\overline{Z}) \subset \widetilde{\psi}^\perp \cap \phi^{-1}T_r{}^\perp \; .$$

Further, since the Levi-Civita connection, ∇^N, is just the tensor product connection on $\overline{T}_r \otimes T_r^\perp$ we see that

$$\nabla_X^N C^\infty(\phi^{-1}\overline{T}_r \otimes \widetilde{\psi}) \subset C^\infty(\phi^{-1}\overline{T}_r \otimes \widetilde{\psi})$$

if and only if

$$\nabla_X C^\infty(\widetilde{\psi}) \subset C^\infty(\widetilde{\psi})$$

and since ∇^N is metric and preserves $T^{(1,0)}\mathbb{G}_r(\mathbb{C}^n)$ it is easy to see that

$$\nabla_X C^\infty(j \circ \psi)^{-1}E^+ \subset C^\infty(j \circ \psi)^{-1}E^+)$$

if and only if

$$\nabla_X C^\infty(\widetilde{\psi}) \subset C^\infty(\widetilde{\psi})$$

from which the theorem follows. $\square$

Corollary 3.4 Let M be a cosymplectic almost Hermitian manifold and $\psi : M \to \mathbb{G}_s(T_r^{\perp})$ a J_2 holomorphic map. Then $\pi_r \circ \psi = \phi : M \to \mathbb{G}_r(\mathbb{C}^n)$ is harmonic and $<\mathrm{Im}\,\partial\phi(X), \mathrm{Im}\,\partial\phi(\overline{Y})> = 0 \;\; \forall X, Y \in T^{(1,0)}M$. This last condition is called *strong conformality* after the strong isotropy of Erdem-Wood [31] in case $\dim M=2$.

A similar analysis may be carried out for the real case: let $T_r \to G_r(\mathbb{R}^n)$ be the tautological bundle and $\psi: M \to F_k(T_r^{\perp})$ a smooth map. We identify ψ with $\widetilde{\psi}$ the bundle of +i eigenspaces of F_ψ on $\phi^{-1}T_r^{\perp\mathbb{C}}$. Thus $\widetilde{\psi}$ is a k-dimensional isotropic sub-bundle of $\phi^{-1}T_r^{\perp\mathbb{C}}$ and we have, by similar arguments:

Theorem 3.5 Let (M,g) be almost Hermitian and $\psi : M \to F_k(T_r^{\perp})$ a smooth map with $\pi_r \circ \psi = \phi : M \to G_r(\mathbb{R}^n)$. Then ψ is f-holomorphic with respect to F_2^r if and only if

$$\mathrm{Im}\,\phi_*(Z) \subset \widetilde{\psi}\;, \quad \text{for all } Z \in T^{(1,0)}M,$$

and

$$\phi^{-1}\nabla_{\overline{Z}}\, C^\infty(\widetilde{\psi}) \subset C^\infty(\widetilde{\psi})\;, \quad \text{for all } Z \in T^{(1,0)}M.$$

Further, ψ is horizontal and holomorphic if and only if, additionally

$$\phi^{-1}\nabla_X\, C^\infty(\widetilde{\psi}) \subset C^\infty(\widetilde{\psi})\;, \;\forall X \in TM.$$

Thus if M is cosymplectic and ψ is J_2 f-holomorphic, ϕ is harmonic and $(\mathrm{Im}\,\phi_*(X), \mathrm{Im}\,\phi_*(Y)) = 0$, for all $X,Y \in T^{(1,0)}M$, where (,) is the complex bilinear extension of the metric on $T_r^{\perp\mathbb{C}}$.

Again we call this last condition strong conformality if $\dim M=2$.

4. *Harmonic maps of surfaces into Grassmanians*

As we have seen in Section 3, J_2 holomorphic maps into $F_k(N,h)$ and $\mathbb{G}_r(T^{(1,0)}N)$ project onto harmonic maps. The fundamental theorem of Eells-Salamon asserts that for 2-dimensional domains, the converse it true and J_2-holomorphic maps into $F_1(N,h)$ (or $\mathbb{G}_1(T^{(1,0)}N)$) can be constructed from conformal harmonic maps into N: for ϕ conformal and harmonic, the corresponding map in $F_1(N,h)$ is essentially the Gauss map.

In this section we give an analogous construction of J_2 holomorphic maps into $F_k(T^{\perp})$ or $\mathbb{G}_r(T^{\perp})$ over certain harmonic maps into Grassmanian.

A. Complex Case

Let M^2 be a Riemann surface with isothermal co-ordinate z, and let $\phi : M^2 \to \mathbb{G}_r(\mathbb{C}^n)$ be a smooth map.

Definition 4.1 ϕ is said to have *(complex) Grassmann order* α if

$$\alpha = \max_{x\in M} \dim \operatorname{Im} \partial\phi\left(\frac{\partial}{\partial z}\right)_x .$$

α is clearly independent of the choice of isothermal co-ordinate.

Theorem 4.2 Let $\phi : M^2 \to \mathbb{G}_r(\mathbb{C}^n)$ be a strongly conformal harmonic map of Grassman order α. Then there exists a unique J_2^r holomorphic map $\psi : M^2 \to \mathbb{G}_\alpha(T_r^{\perp})$ with $\pi_r \circ \psi = \phi$ characterised by

$$\psi(x) = \operatorname{Im} \partial\phi\left(\frac{\partial}{\partial z}\right)_x \quad \text{whenever the right hand side has dimension } \alpha.$$

Proof Consider $\phi^{-1}T_r \to M$, $\phi^{-1}T_r^{\perp} \to M$. These are complex vector bundles over M with connections, ∇, induced from the flat connection on $M \times \mathbb{C}^n$.

We equip these bundles with the Koszul-Malgange complex structures induced by the connections (see theorem B1.1 of Chapter 4). We denote the Levi-Civita connection on $\mathbb{G}_r(\mathbb{C}^n)$ by ∇^N. Now let $\rho_1 \ldots \rho_r$ be a local holomorphic frame for $\phi^{-1}T_r$ and choose an isothermal co-ordinate z on M. We write $\frac{\partial\phi}{\partial z}$ for $\partial\phi(\frac{\partial}{\partial z})$.

Since ϕ is harmonic $\phi^{-1}\nabla^N_{\frac{\partial}{\partial \bar{z}}} \frac{\partial\phi}{\partial z}$ vanishes identically. Thus

$$\nabla_{\frac{\partial}{\partial \bar{z}}}\left(\frac{\partial\phi}{\partial z}(\rho_i)\right) = \left[\phi^{-1}\nabla^N_{\frac{\partial}{\partial \bar{z}}} \frac{\partial\phi}{\partial z}\right](\rho_i) - \partial\phi\left(\nabla_{\frac{\partial}{\partial \bar{z}}} \rho_i\right) = 0 \quad .$$

Thus each $\frac{\partial\phi}{\partial z}(\rho_i)$ is a local holomorphic section of $\phi^{-1}T_r^{\perp}$ and so for each $\{i_1 \ldots i_\alpha\} \subset \{1,\ldots,r\}$, $\Lambda^\alpha_j \frac{\partial\phi}{\partial z}(\rho_{i_j})$ is a holomorphic section of $\Lambda^\alpha\phi^{-1}T_r$ and thus has isolated zeroes or vanishes identically.

Since $\max\dim \operatorname{Im} \frac{\partial\phi}{\partial z} = \alpha$ there exists $\{i_1 \ldots i_\alpha\}$ such that $\Lambda^\alpha_j \frac{\partial\phi}{\partial z}(\rho_{i_j})$ does not vanish identically. At a zero z_o of $\Lambda^\alpha_j \frac{\partial\phi}{\partial z}(\rho_{i_j})$,

$$\Lambda^\alpha_j \frac{\partial\phi}{\partial z}(\rho_{i_j}) = (z-z_o)^k W,$$

where W is a holomorphic section of $\Lambda^\alpha\phi^{-1}T_r^{\perp}$ such that $W(z_o) \neq 0$ and $W(z)$ is decomposible for all z.

W defines a local rank α sub-bundle of $\phi^{-1}T_r^{\perp}$ which coincides with $\operatorname{Im}\frac{\partial\phi}{\partial z}$ off the isolated points where $\dim \operatorname{Im}\frac{\partial\phi}{\partial z} < \alpha$. Thus $\operatorname{Im}\frac{\partial\phi}{\partial z}$ extends to a holomorphic rank α sub-bundle $\tilde{\psi}$ of $\phi^{-1}T_r$ with corresponding map $\psi : M^2 \to \mathbb{G}_\alpha(T_r^{\perp})$.

Obviously

$$\mathrm{Im}\,\frac{\partial\phi}{\partial z}\subset\tilde{\psi}\ ,$$

and by strong conformality

$$\mathrm{Im}\,\frac{\partial\phi}{\partial\bar{z}}\subset\tilde{\psi}^{\perp}\cap\phi^{-1}T_r^{\perp}\ .$$

Lastly since $\tilde{\psi}$ is holomorphic we have $\nabla_{\frac{\partial}{\partial\bar{z}}}C^{\infty}(\tilde{\psi})\subset C^{\infty}(\tilde{\psi})$ whence ψ is J_2 holomorphic by theorem 3.3. □

To get a 1:1 correspondence between harmonic and J_2 holomorphic maps we must identify those J_2-holomorphic maps which project onto order α harmonic maps. This can be done as follows: identify $\mathbb{G}_{\alpha}(T_r^{\perp})$ with $F(r,\alpha,n-\alpha-r;\mathbb{C}^n)$ as in Section 2 and equip it with the Kählerian J_1^r complex structure. Then with notation as in Section 2:

$$T^{(1,0)}F(r,\alpha,n-\alpha-r;\mathbb{C}^n)\cong L(T_r,T_{\alpha})+L(T_{n-r-\alpha},T_r)+L(T_{n-r-\alpha},T_s),$$

where the first two summands are horizontal. It is easy to see that $\pi_r\circ\psi$ has order α if and only if

$$\max_{x\in M}\dim\frac{\partial\psi}{\partial z}(T_r)=\alpha.$$

Call such a ψ *non-degenerate* and we have

Corollary 4.3 There is a 1:1 correspondence between non-degenerate J_2-holomorphic maps $\psi:M^2\to\mathbb{G}_{\alpha}(T_r^{\perp})$ and strongly conformal harmonic maps of Grassmann order α $\phi:M\to\mathbb{G}_r(\mathbb{C}^n)$ given by the construction of theorem 4.4.

Remark For $r = 1$, α is necessarily 1 if ϕ non-constant, strong conformality reduces to conformality and the theorem reduces to that of Eells-Salamon [25] since $\mathbb{G}_1(T_1^{\perp})$ is the same as $\mathbb{G}_1(T^{(1,0)}\mathbb{C}P^{n-1,\mathbb{C}})$.

B. *Real Case*

Let $\phi : M^2 \to G_r(\mathbb{R}^n)$ be a smooth map.

Definition 4.4 ϕ is said to have *real Grassman order* k if

$$k = \max_{x \in M} \dim_{\mathbb{C}} \operatorname{Im} \phi_* \frac{\partial}{\partial z} .$$

Reasoning exactly analogous to that in part A gives

Theorem 4.5 Let $\phi : M^2 \to G_r(\mathbb{R}^n)$ be a strongly conformal harmonic map of Grassman order k. Then there is a unique f-holomorphic map (with respect to F_2^r), $\psi : M^2 \to F_k(T_r^{\perp})$ with $\pi_r \circ \psi = \phi$ characterised by

$$\operatorname{Im} \phi_* \frac{\partial}{\partial z} \subset \widetilde{\psi}.$$

5. *Applications*

The foregoing machinery can be used to produce new harmonic maps from old. Recall that $\mathbb{G}_s(T_r^{\perp})$ may be identified with $F(r,s,n-r-s,\mathbb{C}^n)$.

Proposition 5.1 Let (M,J) be almost Hermitian cosymplectic and $\psi : M \to F(r,s,n-r-s;\mathbb{C}^n)$ be a J_2^r holomorphic map. Then denoting by $\pi_i : F(r,s,n-r-s,\mathbb{C}^n) \to G_i(\mathbb{C}^n)$ $i = r,s,n-r-s$ the projections onto the constituent Grassmannians we have $\pi_r \circ \psi, \pi_s \circ \psi$ and $\pi_{n-r-s} \circ \psi$ are all harmonic.

Further: (i) $\pi_s \circ \psi$ is holomorphic if and only if $\pi_r \circ \psi$ is antiholomorphic,

(ii) $\pi_s \circ \psi$ is antiholomorphic if and only if ψ is π_r-horizontal,

(iii) $\pi_{n-r-s} \circ \psi$ is holomorphic if and only if ψ is π_r-horizontal,

(iv) $\pi_{n-r-s} \circ \psi$ is antiholomorphic iff and only if $\pi_r \circ \psi$ is holomorphic.

Proof By proposition 2.4 the fibrations $\pi_r, \pi_s, \pi_{n-r-s}$ are twistor fibrations for which the J_2 almost complex structures coincide and so the harmonicity of the projections of ψ follows immediately from corollary 3.4. For the rest, since ψ is J_2^r holomorphic

$$\psi_* \frac{\partial}{\partial z} \in \overline{T}_r \otimes T_s + T_r \otimes \overline{T}_{n-r-s} + \overline{T}_s \otimes T_{n-r-s},$$

and we have

$$\pi_{r*}\,\overline{T}_r \otimes T_s \subset T^{(1,0)}\mathbb{G}_r(\mathbb{C}^n)\ ,\ \pi_{s*}(\overline{T}_r \otimes T_s) \subset T^{(0,1)}\mathbb{G}_s(\mathbb{C}^n)\ ,$$

$$\pi_{r*}\,T_r \otimes \overline{T}_{n-r-s} \subset T^{(0,1)}\mathbb{G}_r(\mathbb{C}^n)\ ,\ \pi_{s*}(T_r \otimes \overline{T}_{n-r-s}) = \{0\}\ ,$$

$$\pi_{r*}\,\overline{T}_s \otimes T_{n-r-s} = \{0\}\qquad ,\ \pi_{s*}(\overline{T}_s \otimes T_{n-r-s}) \subset T^{(1,0)}\mathbb{G}_s(\mathbb{C}^n).$$

Thus $\pi_s \circ \psi$ is holomorphic if and only if $\psi_* \frac{\partial}{\partial z}$ has no $\overline{T}_r \otimes T_s$ component or, equivalently, $\pi_r \circ \psi$ is antiholomorphic. The rest of the proof is similar. □

Thus:

Theorem 5.2 Let $\phi : M^2 \to \mathbb{G}_r(\mathbb{C}^n)$ be strongly conformal harmonic of Grassmann order $\alpha > 0$, and let $\psi : M^2 \to \mathbb{G}_s(T_r^{\perp})$ be the J_2^r holomorphic lift of ϕ. Then

(i) ψ is harmonic with respect to any of the Kähler metrics, associated to J_1^r, J_1^s or J_1^{n-r-s},

(ii) $\pi_s \circ \psi$, $\pi_{n-r-s}\psi$ are harmonic and strongly conformal,

(iii) if ψ is not horizontal, $\pi_s \circ \psi$ is neither holomorphic nor antiholomorphic.

Proof (i) follows from the fact that $J_2^s = J_2^r = J_2^{n-r-s}$ satisfy condition A by theorem 2.3,

(ii) is immediate from 5.1, the strong conformality following from corollary 3.4,

(iii) is immediate from 5.1 since if $s > 0$, ϕ is not antiholomorphic.

□

Remarks (i) Almost everywhere we have

$$\pi_s \circ \psi = \operatorname{Im} \frac{\partial\phi}{\partial z} ,$$

$$\pi_{n-r-s} \circ \psi = (\phi + \operatorname{Im} \frac{\partial\phi}{\partial z})^{\perp}$$

and so these projections may be thought of as generalised Gauss maps.

(ii) Even if ϕ is holomorphic, $\pi_s \circ \psi$ is not $\pm$ holomorphic unless ψ is horizontal. For r=1, it is easy to show that ϕ holomorphic and ψ horizontal implies that ϕ is totally geodesic (c.f. Chapter 7). Thus non-totally geodesic holomorphic maps into $\mathbb{C}P^{n-1}$ give rise to non-$\pm$ holomorphic harmonic maps into $\mathbb{C}P^{n-1}$ and, of course, into $F(1,1,n-2,\mathbb{C}^n)$ equipped with its Kählerian J_1 structure.

(iii) Indeed, let M^{2m} be any Kähler manifold and $\phi : M \to \mathbb{C}P^{n-1}$ a non-totally geodesic holomorphic immersion. Then $\operatorname{Im} \partial\phi(T^{(1,0)}M)$ defines a lift $\psi: M^{2m} \to \mathbb{G}_m(T_1^{\perp})$ which is J_2 holomorphic and non-horizontal (the key point being that ϕ is harmonic with $\beta^{(1,1)}$ vanishing, where β is the second fundamental form of ϕ) and so we have harmonic non-holomorphic maps into $\mathbb{G}_m(\mathbb{C}^n)$ and $F(1,m,n-m-1;\mathbb{C}^n)$. These remarks provide an interpretation of a result of Ishihara [44] in our context.

(iv) Let $\psi : M^2 \to \mathbb{G}_s(T_r^{\perp})$ be holomorphic and horizontal then $\pi_s \circ \psi$, $(\pi_{n-r-s} \circ \psi)^{\perp}$ are antiholomorphic and in fact form a ∂'' pair of antiholomorphic vector bundles in the sense of Erdem-Wood. This observation shows how to interpret their results [31] in our context.

(v) Under suitable non-degeneracy conditions on ψ it is clear that ψ will be the common holomorphic lift of $\pi_r \circ \psi$, $\pi_s \circ \psi$ and $\pi_{n-r-s} \circ \psi$.

(vi) Changing the orientation on M will not affect the harmonicity of any of the maps under consideration, so by considering $\mathrm{Im}\frac{\partial\phi}{\partial \bar{z}}$ and so on, we can construct yet more harmonic maps into various Grassmannians and flag manifolds (in general different ones!).

We end this chapter with an application of the real version of these ideas to prove a theorem of Obata in our contex.

Let $\phi : M^2 \to S^{n-1}$ be a conformal immersion, the *normal Gauss map* of ϕ, $\tilde{\phi} : M^2 \to G_{n-3}(\mathbb{R}^n)$ is given by

$$\tilde{\phi}(x) = (\phi(x) + \mathrm{Im}\, d\phi)^{\perp}.$$

Theorem 5.3 (Obata [53]) ϕ is harmonic if and only if $\tilde{\phi}$ is harmonic.

Proof Recall the isomorphism of $F_1(T_1^{\perp})$ with $F_1(T_{n-2}^{\perp})$ in Section 2. If ψ is the lift of ϕ then it is easy to see that its projection onto $G_{n-2}(\mathbb{R}^n)$ via this identification is precisely $\tilde{\phi}$. Thus since by proposition 2.2 ψ is J_2 antiholomorphic as a map into $F_1(T_{n-2}^{\perp})$, $\tilde{\phi}$ is harmonic and strongly conformal. Further, a calculation yields that $\mathrm{Im}\, d\tilde{\phi}(TM) \perp \phi$ so that if $\tilde{\phi}$ is non-constant and harmonic, the projection of its lift back onto S^{n-1} is ϕ.

Lastly, if $\tilde{\phi}$ is constant, ϕ is clearly totally geodesic and hence harmonic. □

APPENDIX

A1. *Adjoint orbits of a compact Lie group* (see [59])

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. We equip $\mathfrak{g}$ with an Ad(G) invariant inner product (e.g. the negative of the Killing form).

Let $\xi_o \in \mathfrak{g}$ and let $C(\xi_o)$ denote the orbit of ξ_o under the adjoint action of G. Then if $H = \{g \in G : \mathrm{Ad}g \cdot \xi_o = \xi_o\}$ is the isotropy subgroup of ξ_o, $C(\xi_o)$ is a homogeneous space $\frac{G}{H}$.

The Lie algebra h of H is given by $h = \{\eta \in \mathfrak{g} : [\xi_o, \eta] = 0\} = \ker \mathrm{ad}(\xi_o)$ and since $\mathrm{ad}(\xi_o)$ is skew-symmetric on $\mathfrak{g}$, $m = \mathrm{Im}\,\mathrm{ad}(\xi_o)$ is an Ad(H)-invariant complement to h in $\mathfrak{g}$. Thus the tangent space of $C(\xi_o)$ at ξ_o is isomorphic to m.

Since $\mathrm{ad}(\xi_o)$ is skew-symmetric, it has only purely imaginary eigenvalues and we denote by $\mathfrak{g}^\lambda$ the $i\lambda$-eigenspace of $\mathrm{ad}(\xi_o)$ in $\mathfrak{g}^{\mathbb{C}}$.

We have:

Lemma 1.1 [59] $\mathfrak{g}^o = h^{\mathbb{C}}$, $\mathfrak{g}^{-\lambda} = \overline{\mathfrak{g}^\lambda}$ and $[\mathfrak{g}^\lambda, \mathfrak{g}^\mu] \subset \mathfrak{g}^{\lambda+\mu}$. Further $\mathrm{Ad}(H)\mathfrak{g}^\lambda \subset \mathfrak{g}^\lambda$.

Thus putting $m^+ = \sum_{\lambda>0} \mathfrak{g}^\lambda$, $m^- = \sum_{\lambda<0} \mathfrak{g}^\lambda$ we have $m^{\mathbb{C}} = m^+ + m^-$, $\overline{m^+} = m^-$, $\mathrm{Ad}(H)m^+ \subset m^+$ and $[m^+, m^+] \subset m^+$ and so we have defined a G-invariant complex structure on $C(\xi_o)$. In fact, if B denotes the negative of the Killing form of $\mathfrak{g}$, the inner product a on m given by

$$a(\xi, \eta) = B(\xi, |\mathrm{ad}\xi_o|\eta)$$

induces a Kähler metric on $C(\xi_o)$. (see below.)

Now let V be a euclidean space, O(V) its orthogonal group with Lie algebra o(V) of skew-symmetric linear transformations.

We identify $o(V)$ with $\Lambda^2 V$ by

$$(u \wedge v)x = (u,x)v - (v,x)u,$$

where $(\ ,\ )$ is the inner product on V.

Then

$$\operatorname{Ad} g\,(u \wedge v) = gu \wedge gv\ ,\ g \in O(V)$$

and

$$\operatorname{ad}A(u \wedge v) = Au \wedge v + u \wedge Av,\quad A \in o(V).$$

Now let $F \in o(V)$ be a rank K f-structure on V. Thus putting $G = O(V)$, we have $F_K(V) = C(F)$.

Denote by V_F^+, V_F^-, V_F^o the $+i,-i,0$ eigenspaces of F in $V^{\mathbb{C}}$. Then it is easy to see that adF has eigenvalues $\pm 2i, \pm i, 0$ in $o(V)^{\mathbb{C}}$ with

$$o(V)^1 = V_F^+ \otimes V_F^o$$
$$o(V)^2 = \Lambda^2 V_F^+$$
$$o(V)^o = V_F^+ \otimes V_F^- + \Lambda^2 V_F^o\ .$$

Lemma 1.2 [59] Let $A \in o(V)$. Then $A \in T_F^{(1,0)} F_K(V)$ if and only if $AV_F^+ = 0$, $AV_F^o \subset V_F^+$ and $AV_F^- \subset V_F^o + V_F^+$.

A2. *Twistor bundles associated to principal G-bundles*

The construction of $F_K(N,h) \to N$ may be generalised as follows [54,59]: Let $P \to N$ be a principal G-bundle over a Riemannian manifold (N,h) such that TN is associated to P via some orthogonal representation $\rho : G \to O(V)$ of G.

Let G act holomorphically on a complex manifold Y and suppose that there is a map $j_o : Y \to F_K(V)$ which is G-equivariant and holomorphic. Let α be a connection on P that induces the Levi-Civita connection on N, and consider $\pi_1 : Z \to N$ where $Z = P \times_G Y$. As before α induces a horizontal distribution on Z which can be identified with $\pi_1^{-1}TN$. Further j_o induces a fibre map $j : Z \to F_K(N,h)$ and so $\pi_1^{-1}TN$ acquires a tautological f-structure, F, given by

$$F = j(z) \text{ on } T_{\pi_1(z)}N, \ z \in Z.$$

Thus, as before, we may define f-structures F_1^z, F_2^z and it is clear that we have

Proposition 2.1 The map $j : Z \to F_K(N,h)$ preserves horizontal distributions and is f-holomorphic with respect to (F_1^z, F_1) and (F_2^z, F_2).

In particular, let Y be a reductive homogeneous space $\frac{G}{H}$ with invariant complex structure, then Z can be identified with $\frac{P}{H}$ and thus P is a principal H-bundle over Z.

Now let $g^{\mathbb{C}} = h^{\mathbb{C}} + m^+ + m^-$ be the decomposition of the Lie algebra of g associated to $\frac{G}{H}$ and let $j_o : \frac{G}{H} \to F_K(V)$ be an equivariant map.

Lemma 2.2 Let $j_o(eH) = F_o$. Then j_o is holomorphic if and only if

$$[F_o, d\rho(\xi)] \in \Lambda^2 V_{F_o}^+ + V_{F_o}^+ \otimes V_{F_o}^o \ , \quad \text{for all } \xi \in m^+ \ .$$

Proof It suffices to show holomorphicity at the identity coset by equivariance. Identifying $F_K(V)$ with $\frac{O(V)}{H_{F_o}}$, where H_{F_o} is the isotropy of F_o, we have a commuting diagram

$$\begin{array}{ccc} G & \xrightarrow{\rho} & O(V) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ \frac{G}{H} & \xrightarrow[j_o]{} & \frac{O(V)}{H_{F_o}} \end{array} \ .$$

Identifying $T_{eH}\frac{G}{H}$ with m via $d\pi_1$, for $\xi \in m^+$ we have

$$dj_o(\xi) = d\pi_2 d\rho(\xi).$$

Thus j_o is holomorphic if and only if

$$d\pi_2 d\rho(\xi) \in T^{(1,0)}_{F_o} F_k(V)$$

or equivalently

$$d\rho(\xi) \in \Lambda^2 V^+_{F_o} + V^+_{F_o} \otimes V^o_{F_o} + h_{F_o}$$

which last is equivalent to

$$[F_o, d\rho(\xi)] \in \Lambda^2 V^+_{F_o} + V^+_{F_o} \otimes V^o_{F_o}. \qquad \square$$

For instance let N be a Kähler manifold and U(N) its U(n)-bundle of unitary frames. Then the Levi-Civita connection of N is a connection on U(N).

Let (V,J) be the standard U(n) module and take Y to be $\mathbb{G}_r(V)$ the Grassmannian of complex r-planes in V (i.e. 2r-real dimensional J-invariant subspaces). U(n) acts transitively on $\mathbb{G}_r(V)$ and $j_o : \mathbb{G}_r(V) \to F_n(V)$ given by

$$\begin{aligned} j_o(W) &= J \text{ on } W \\ &= -J \text{ on } W^\perp \end{aligned}$$

is a U(n) equivariant map.

Fixing $W \in \mathbb{G}_r(V)$, we identify $\mathbb{G}_r(V)$ with the hermitian symmetric space $\frac{U(n)}{U(r)\times U(n-r)}$. Identifying u(n) with $\Lambda^{1,1}V^{\mathbb{C}}$, the tangent space at W is isomorphic to $\overline{W} \otimes W^\perp + W \otimes \overline{W}^\perp$. We choose the complex structure on $G_r(V)$ with +i-eigenspace at W given by $W \otimes \overline{W}^\perp$. Then if $\xi \in W \otimes \overline{W}^\perp$, since ρ is just the inclusion $U(n) \hookrightarrow O(n)$ we have

$$[j_o(W), \xi] = 2i\xi$$

and putting $j_o(W) = J_o$, $V^+_{J_o} = W + \overline{W}^{\perp}$, so that $[j_o(W), \xi] \in \Lambda^2 V^+_{J_o}$. Thus by Lemma 2.2 j_o is holomorphic. If we identify $U(N) \times_{U(n)} \mathbb{G}_r(V)$ with $\mathbb{G}_r(T^{(1,0)}N)$ we see that the induced map $j: \mathbb{G}_r(T^{(1,0)}N) \to J(N)$ is precisely that described in Section 1 and thus theorem 1.2 of that section is proved.

A3. *Twistor bundles over symmetric spaces*

Let $N = \frac{G}{H}$ be a Riemannian symmetric space where G is compact and the metric on N is that induced by the negative of the Killing form on g. We have the canonical decomposition of the Lie algebra of G:

$$g = h + m \text{ with } [h,m] \subset m \;,\; [m,m] \subset h.$$

Let $\xi_o \in h$ and suppose that $\mathrm{ad}\xi_o$ is a rank k f-structure when restricted to m.* Let h^λ, m^λ denote the $i\lambda$-eigenspaces of $\mathrm{ad}\xi_o$ in h and m and put $h^+ = \sum_{\lambda > 0} h^\lambda$, $m^+ = m^1$ etc.

Denote by $C_H(\xi_o)$ the orbit of ξ_o in h under Ad(H), then, as in Section A1, the isotropy subgroup of ξ_o in H, denoted K, has Lie algebra h^o and $C_H(\xi_o) = \frac{H}{K}$ has an invariant complex structure with +i eigenspace at ξ_o given by h^+.

Lemma 3.1 Define $j_o : C_H(\xi_o) \to F_k(m)$ by

$$j_o(\eta) = \mathrm{ad}\eta.$$

Then j_o is H-equivariant and holomorphic.

* For examples of such ξ_o see pp 96-97

Proof Let $v \in m$, then $j_o(\mathrm{Ad}h\eta)v = [\mathrm{Ad}(h)\eta, v]$

$$= [\mathrm{Ad}(h)\eta,\ \mathrm{Ad}(h)\mathrm{Ad}(h^{-1})v]$$
$$= \mathrm{Ad}(h)\mathrm{ad}\eta\ \mathrm{Ad}(h^{-1})v$$
$$= \mathrm{Ad}(h)\ j_o(\eta)\ \mathrm{Ad}(h)^{-1}v.$$

Thus j_o is equivariant.

In this case j_o is induced from the adjoint representation of H on m so if $\xi \in h^+$,

$$[j_o(\xi_o),\ d\mathrm{Ad}(\xi)] = [\mathrm{ad}\xi_o, \mathrm{ad}\xi] = \mathrm{ad}[\xi_o, \xi].$$

Now by Lemma 1.1, $[\xi_o, \xi] \in h^+$ and

$$[h^+, m^1] = 0,\ [h^+, m^o] \subset m^1\ [h^+, m^{-1}] \subset m^o + m^1$$

so by Lemmas 1.2 and 2.2 we conclude that j_o is holomorphic. □

Now $\pi : G \to \frac{G}{H}$ is a principal H-bundle and $G\times_H m$ is isomorphic to TN via $g \cdot \xi = \pi_* L_{g*}\xi$. Further since $\frac{G}{H}$ is symmetric the canonical connection, α, on G induces the Levi-Civita connection on $\frac{G}{H}$. Thus the construction of A2 yields f-structures on $\frac{G}{K}$ and an f-holomorphic map

$$\begin{tikzcd} j : \frac{G}{K} \arrow[rr] \arrow[dr] & & F_k(\frac{G}{H}) \arrow[dl] \\ & \frac{G}{H} & \end{tikzcd}$$

In fact the f-structures are easy to identify in this case. Chasing through the identifications shows that F_1^z, F_2^z are both G-invariant and have +i eigenspaces at the identity coset given by

$$F_1^z = i \text{ on } h^+ + m^1$$

$$F_2^z = i \text{ on } h^- + m^1 \ .$$

As an application of the foregoing, let $G = SO(n)$ and $H = SO(r) \times SO(n-r)$. Thus $\frac{G}{H}$ is isometric to $G_r(\mathbb{R}^n)$. Let $W \in G_r(\mathbb{R}^n)$ correspond to the identity coset, then under the usual identification we have

$$o(n) = o(r) + o(n-r) + W \otimes W^{\perp}$$

for the canonical decomposition of $o(n)$.

Let $F_o \in o(n-r)$ be a rank k f-structure on $W^{\perp}$.
Then $\mathrm{ad}F = \mathrm{id} \otimes F$ on $W \otimes W^{\perp}$ is an f-structure on $W \otimes W^{\perp}$ and the stabiliser of F in $SO(r) \times SO(n-r)$ is isomorphic to $SO(r) \times U(k) \times SO(n-r-2K)$, thus we have f-structures and an f-holomorphic map j

$$\begin{tikzcd} j : \dfrac{SO(n)}{SO(r) \times U(k) \times SO(n-r-2k)} \arrow[r] \arrow[dr] & F_k(G_r(\mathbb{R}^n)) \arrow[d] \\ & G_r(\mathbb{R}^n) \ . \end{tikzcd}$$

Further $C_H(F_o)$ is precisely $F_k(W^{\perp})$ for $2k < n-r$ (it is $J_o(W^{\perp})$ if $2k = n-r$) and since the tautological bundle T on $G_r(\mathbb{R}^n)$ is naturally isomorphic to $SO(n) \times_{SO(r) \times SO(n-r)} W$ we have an isomorphism between $\frac{G}{K}$ and $F_k(T^{\perp})$ under which j is given by

$$j(F) = \mathrm{id} \otimes F \ .$$

Further, if $W_{F_o}^{\perp +}$, $W_{F_o}^{\perp -}$, $W_{F_o}^{\perp o}$ denote the eigenspaces of F_o it is easy to see that

$$m^+ = W \otimes W_{F_o}^{\perp +} \ , \ h^+ = \Lambda^2 W_{F_o}^{\perp +} + W_{F_o}^{\perp +} \otimes W_{F_o}^{\perp o}$$

from which follow the identification of the eigenspaces of F_1^r, F_2^r in terms of tautological bundles given in Section 1. Thus the proof of the first part of theorem 2.1 is complete.

Remark For $2k = n-r$, to treat $F_k(T^{\perp})$ we must replace $SO(n)$ by $O(n)$ and argue as above.

Now putting $G = U(n)$, $H = U(r) \times U(n-r)$ we can repeat the argument. Letting $W \in \mathbb{G}_r(\mathbb{C}^n)$ correspond to the identity coset we have

$$u(n) = u(r) + u(n-r) + (\overline{W} \otimes W^{\perp} + W \otimes \overline{W^{\perp}}) \cap u(n) .$$

Let $J_o \in u(n-r)$ be an almost complex structure on $W^{\perp}$ which agrees with the ambient complex structure on an s-dimensional subspace $V \subset W^{\perp}$. The stabiliser of J_o is isomorphic to $U(r) \times U(s) \times U(n-r-s)$ and $C_H(J_o)$ is naturally isomorphic to $\mathbb{G}_s(W^{\perp})$. Proceeding as above provides the proof of theorem 2.3.

A4. *Properties of Twistor Bundles*

Let $Z = P \times_G Y \to N$ be a bundle of almost complex structures constructed as in A2 with fibre map $j : Z \to J(N)$ which we assume to be an immersion. We have

Proposition 4.1 The complex structure J_2^Z is not integrable.

Proof If J_2^Z is integrable, then $j(Z)$ is a complex, non-horizontal submanifold of $J(N)$ on which J_2 is integrable contradicting a proposition of Salamon [61].

In general, the f-structures on $F_K(N,h)$ or $\mathbb{G}_r(T^{(1,0)}N)$ are not very well behaved requiring stringent conditions on the curvature of N for integrability of F_1 or for F_2 to satisfy condition A. (See Rawnsley [59], Eells-Salamon [25], Hitchin [43] and Chapter 7.) However, the situation for the twistor bundles over symmetric spaces constructed in A3 is much better.

Theorem 4.2 Let $\frac{G}{H}$ be a Riemannian symmetric space and $Z = \frac{G}{K}$ a twistor bundle over $\frac{G}{H}$ constructed as in A3. Then F_1 is integrable and there exists a metric on Z for which

(i) $\pi : Z \to \frac{G}{H}$ is a Riemannian submersion,

(ii) F_2 satisfies condition A,

(iii) If F_1 is a complex structure, (Z, F_1) is a Kähler manifold. (Compare [43].)

Proof It suffices to check everything at the identity coset by homogeneity.

Let Z arise from $C_H(\xi_o)$, $\xi_o \in h$, then at eK, F_1^Z has +i eigenspace $h^+ + m^1$ and

$$[h^+ + m^1, h^+ + m^1] \subset h^+$$

by Lemma 1.1. Thus F_1^Z is integrable.

Now define an inner product on $\mathrm{ad}(\xi_o)\, h + m$ by

$$\begin{aligned} a(\xi,\eta) &= (\xi, |\mathrm{ad}(\xi_o)|\eta) \quad \text{for } \xi,\eta \in \mathrm{ad}(\xi_o)h \\ &= (\xi,\eta) \qquad\qquad \text{for } \xi,\eta \in m \text{ and zero otherwise,} \end{aligned}$$

where (,) is the negative of the Killing form.

a is clearly AdK invariant and so gives rise to an invariant metric on Z for which π is a Riemannian fibration.

Recall that $G \to \frac{G}{K}$ is a principal K-bundle and let D denote the canonical connection on G. Since F_1^z, F_2^z are G invariant, $DF_1^z = DF_2^z = 0$ (see [45, Vol.II Ch.X]) and at eK, the torsion T of D is given by

$$a(T(\xi,\eta),\delta) = -a([\xi,\eta],\delta) \quad , \quad \xi,\eta,\delta \in ad(\xi_o)h + m. \tag{1}$$

Now let ∇ denote the Levi-Civita connection of (Z,a). A standard calculation (see [59]) yields

$$a(\nabla_X Y,Z) = a(D_X Y,Z) + \tfrac{1}{2}[a(T(X,Y),Z) - a(T(Y,Z),X) + a(T(Z,X),Y)].$$

Now let $X,Y,Z \in T_{eK}Z$ and suppose that each vector lies in a single eigenspace of $ad\xi_o$ in either m or h.

For $X \in h$ put $|X| = |\lambda|$

For $X \in m$ put $|X| = 1$.

Then using (1) and the definition of a together with the fact that ad(X) is skew with respect to (,) for any X in g we have

$$a(\nabla_X Y,Z) = a(D_X Y,Z) - \tfrac{1}{2}(|Y| + |Z| - |X|)([Y,Z],X).$$

Now let $Y,Z \in h^+ + m^1$, then $([Y,Z],X)$ is only non-zero if $X \in h^{-|Y|-|Z|}$ in which case $|Y| + |Z| - |X| = 0$. Thus

$$a(\nabla Y,Z) = a(DY,Z) = 0 \quad \text{for} \quad Y,Z \in h^+ + m^1$$

thus establishing that a is Kählerian for F_1^z if $m^o = \{0\}$.

Similar arguments using $[h^+,m^1] = 0$ and $[m,m] \subset h$ establish the fact that F_2 satisfies condition A. □

CHAPTER 7

ISOTROPIC HARMONIC MAPS AND HORIZONTAL HOLOMORPHIC MAPS

The classifications of Calabi [13], Eells-Wood [29] and Erdem-Wood [31] consider harmonic maps which are isotropic in the sense of Chapter 4 and associate to them holomorphic maps into complex manifolds. In our context, the isotropy condition is precisely what guarantees that the twistor lift is horizontal and thus holomorphic with respect to the (sometimes) integrable J_1 complex structure.

Thus, in this chapter, we turn to isotropic harmonic maps of surfaces and attempt to classify them by horizontal f-holomorphic maps into twistor bundles. It will be seen that a really satisfactory theory is obtained only in case that the target manifold is a space form. Thus our theorem may be viewed as an abstract version of Calabi's theorem.

1. *Horizontal holomorphic maps of surfaces into twistor bundles*

Let $\phi: M^2 \to (N,h)$ be a smooth map of a Riemann surface into Riemannian manifold. We recall the following definitions from Chapter 4.

Definition 1.1 i) ϕ is *real isotropic* if

$$(\nabla^{\alpha}_{\frac{\partial}{\partial z}} \phi_* \frac{\partial}{\partial z}, \nabla^{\beta}_{\frac{\partial}{\partial z}} \phi_* \frac{\partial}{\partial z}) \equiv 0 \qquad \text{for all } \alpha,\beta \geq 0 \text{ and any}$$

isothermal co-ordinate on M. Here $\nabla^{\alpha}_{\frac{\partial}{\partial z}} = \nabla_{\frac{\partial}{\partial z}} \circ \ldots \circ \nabla_{\frac{\partial}{\partial z}}$, α times, ∇ is the pullback connection on $\phi^{-1}TN^{\mathbb{C}}$ and (,) the complex bilinear extension of the inner product on $\phi^{-1}TN$.

ii) If N is a Kahler manifold, denote by $<,>$ the Hermitian inner product on $\phi^{-1}TN^{(1,0)}$ and let $\frac{\partial\phi}{\partial z}, \frac{\partial\phi}{\partial \bar{z}}$ denote the projections of $\phi_*\frac{\partial}{\partial z}, \phi_*\frac{\partial}{\partial \bar{z}}$ onto $T^{(1,0)}N$. Then ϕ is *complex isotropic* if

$$<\nabla^{\alpha}_{\frac{\partial}{\partial z}} \frac{\partial\phi}{\partial z}, \nabla^{\beta}_{\frac{\partial}{\partial \bar{z}}} \frac{\partial\phi}{\partial \bar{z}}> \equiv 0 \quad \text{for all } \alpha,\beta \geq 0 .$$

In the sequel we will mostly be concerned with real isotropy.

Proposition 1.1 Let $\psi : M^2 \to F_k(N,h)$ be horizontal and f-holomorphic (equivalently: f-holomorphic with respect to F_1 and F_2). Then $\phi = \pi_o\psi : M \to N$ is harmonic and real isotropic.

Proof By theorem 3.1 of Chapter 6, ψ is horizontal and f-holomorphic if and only if

$$\phi_*\frac{\partial}{\partial z} \in \psi^{-1}E^+$$

and

$$\nabla_X\, C^{\infty}(\psi^{-1}E^+) \subset C^{\infty}(\psi^{-1}E^+) \qquad \text{for all } X \in TM .$$

In particular $\nabla^{\alpha}_{\frac{\partial}{\partial z}}\, \phi_*\frac{\partial}{\partial z}$ is a local section of $\psi^{-1}E^+$ for all $\alpha \geq 0$ and the isotropy of ϕ follows immediately from the fact that $\psi^{-1}E^+$ is an isotropic subbundle of $TN^{\mathbb{C}}$. The harmonicity of ϕ is a consequence of theorem 3.2 of Chapter 6 (or observe that $\nabla_{\frac{\partial}{\partial \bar{z}}}\phi_*\frac{\partial}{\partial z}$ is a real section of $\overline{\psi^{-1}E^+}$ and must therefore vanish).

Remark As we restrict attention to subbundles of $F_k(N,h)$ the conditions for horizontality become more stringent. A similar argument to the above shows that a horizontal holomorphic map into $\mathbb{G}_r(T^{(1,0)}N)$ over a Kahler manifold has complex isotropic projection, while a horizontal holomorphic map into $G_s(T_r^{\perp})$ over $G_r(\mathbb{C}^n)$ has a strongly isotropic projection in the sense of Erdem-Wood [31], that is

$$<\mathrm{Im}\ \nabla^{\alpha}_{\frac{\partial}{\partial z}} \frac{\partial\phi}{\partial z}, \mathrm{Im}\ \nabla^{\beta}_{\frac{\partial}{\partial \bar{z}}} \frac{\partial\phi}{\partial \bar{z}}> \equiv 0 \qquad \text{for all } \alpha,\beta \geq 0 .$$

In some cases the horizontality condition is very restrictive.

Proposition 1.2 Let (M,J) be a 2m-dimensional almost Hermitian manifold and $\psi: M \to F_m(N,h)$ a horizontal, f-holomorphic map such that $\phi = \pi \circ \psi$ is an immersion. Then, with respect to the metric ϕ^*h on M, ϕ is totally geodesic.

Proof Let $Z_1, \ldots, Z_m$ be a frame for $T^{(1,0)}M$. Then $\{\phi_* Z_i\}_{i=1}^m$ spans $\psi^{-1}E^+$ and $\nabla_X \phi_* Z_i \in C^{\infty}(\psi^{-1}E^+)$ for all $X \in TM$. Thus

$$\nabla_X \phi_* Y \in \phi_*(TM)$$

for all $X,Y \in TM$ and so $\nabla d\phi$ takes values in $\phi_*(TM)$. However, if ϕ is isometric, $\nabla d\phi$ only takes values normal to $\phi_*(TM)$ and thus must vanish identically. So with respect to ϕ^*h, ϕ is totally geodesic. □

Corollary 1.3 Let dim $N = 3$ and $\phi: M^2 \to (N,h)$ be an isotropic minimal immersion of a Riemann surface. Then ϕ is totally geodesic.

Proof Define $\psi: M^2 \to F_1(N,h)$ by span $\{\phi_* \frac{\partial}{\partial z}\} = \psi^{-1}E^+$, this is the Gauss lift of Eells-Salamon [25]. ψ is an f-holomorphic horizontal map and the result follows from proposition 1.2. Alternatively a direct calculation shows that the isotropy condition implies that there is no normal component of $\nabla_{\frac{\partial}{\partial z}} \phi_* \frac{\partial}{\partial z}$ since the normal bundle is 1-dimensional. □

2. *Isotropic harmonic maps of surfaces*

We now construct horizontal f-holomorphic maps from isotropic harmonic maps into a space form.

Notation Let $\phi\colon M^2 \to (N,h)$ be a smooth map and z an isothermal co-ordinate on M^2. Denote $\phi_* \frac{\partial}{\partial z}$ by $\partial\phi$ and let

$$\partial^\alpha \phi = \nabla_{\frac{\partial}{\partial z}} \circ \ldots \circ \nabla_{\frac{\partial}{\partial z}} \phi_* \frac{\partial}{\partial z} \qquad \text{where } \nabla_{\frac{\partial}{\partial z}} \text{ is iterated } (\alpha - 1) \text{ times.}$$

Let W_ϕ^α be the subset of $\phi^{-1}TN^{\mathbb{C}}$ whose fibre at $x \in M$ is given by

$$W_{\phi,x}^\alpha = \text{span}\,\{\partial\phi, \partial^2\phi, \ldots, \partial^\alpha\phi\} \qquad \text{for } \alpha \geq 1,$$

$$W_{\phi,x}^0 = \{0\} \quad .$$

Note that this is not in general a subbundle.

Lemma 2.1 Let $\phi\colon M^2 \to (N,h)$ be an isotropic harmonic map and suppose that (N,h) has constant sectional curvatures. Then for all $\alpha \geq 1$

$$\nabla_{\frac{\partial}{\partial \bar z}} \partial^\alpha \phi \in W_\phi^{\alpha-1}$$

whence $\nabla_{\frac{\partial}{\partial \bar z}} (\partial\phi \wedge \ldots \wedge \partial^\alpha \phi)$ vanishes identically for all $\alpha \geq 1$.

Proof The argument is that of theorem B.2.4(i) of Chapter 4 which for convenience we reproduce here.

We induct on α, the case $\alpha=1$ following immediately from the harmonicity of ϕ.

Now suppose $\nabla_{\frac{\partial}{\partial\bar{z}}} \partial^k\phi \in W_\phi^{k-1}$ for $k < \alpha$.

$$\nabla_{\frac{\partial}{\partial\bar{z}}} \partial^\alpha\phi = \nabla_{\frac{\partial}{\partial\bar{z}}} \nabla_{\frac{\partial}{\partial z}} \partial_\phi^{\alpha-1} = \nabla_{\frac{\partial}{\partial z}} \nabla_{\frac{\partial}{\partial\bar{z}}} \partial_\phi^{\alpha-1} + \phi^*R^N(\frac{\partial}{\partial\bar{z}}, \frac{\partial}{\partial z})\partial_\phi^{\alpha-1}$$

The first summand of the right hand side is contained in $W_\phi^{\alpha-1}$ by the induction hypothesis and so it remains to prove that

$$\phi^*R^N(\frac{\partial}{\partial\bar{z}}, \frac{\partial}{\partial z})\partial^{\alpha-1}\phi \in W_\phi^{\alpha-1}$$

where R^N is the curvature tensor of (N,h). Now by the curvature hypothesis

$$\begin{aligned}\phi^*R^N(\frac{\partial}{\partial\bar{z}}, \frac{\partial}{\partial z})\partial^{\alpha-1}\phi &= R^N(\overline{\partial\phi}, \partial\phi)\partial^{\alpha-1}\phi \\ &= c[(\overline{\partial\phi},\partial^{\alpha-1}\phi)\partial\phi - (\partial\phi, \partial^{\alpha-1}\phi)\overline{\partial\phi}]\end{aligned}$$

which is contained in $W_\phi^{\alpha-1}$ since the second summand vanishes by isotropy. This completes the induction.

The second assertion of the lemma now follows since

$$\nabla_{\frac{\partial}{\partial\bar{z}}}(\partial\phi\wedge\ldots\wedge\partial^\alpha\phi) = \sum_{i=1}^{\alpha} \partial\phi\wedge\ldots\wedge \nabla_{\frac{\partial}{\partial\bar{z}}} \partial^i\phi\wedge\ldots\wedge\partial^\alpha\phi$$

and for each i

$$\partial\phi\wedge\ldots\wedge \nabla_{\frac{\partial}{\partial\bar{z}}} \partial^i\phi \equiv 0 \quad . \qquad \square$$

(cf. Wood [75])

Definition Let $\phi\colon M \to (N,h)$ be a smooth map. The ∂-*order* of ϕ is the largest integer k such that

$$\partial\phi\wedge\ldots\wedge\partial^k\phi \not\equiv 0 .$$

Equivalently it is given by

$$k = \max_{x\in M,\alpha\in\mathbb{Z}^+} \dim W^{\alpha}_{\phi,x} .$$

Clearly k is independent of the choice of isothermal co-ordinate since W^{α}_{ϕ} is.

Theorem 2.2 Let $\phi\colon M^2 \to (N,h)$ be an isotropic harmonic map of ∂-order k from a connected Riemann surface into a space form N. Then there is a unique horizontal f-holomorphic map $\psi\colon M^2 \to F_k(N,h)$ such that $\pi_o\psi = \phi$ which is characterised by

$$\tilde{\psi}_x = W^k_{\phi,x}$$

at all points where the right hand side has dimsion k.

Proof We equip $\Lambda^k\phi^{-1}TN^{\mathbb{C}}$ with its Koszul-Malgrange complex structure relative to the pull-back of the Levi-Civita connection ∇ on N. Then by Lemma 2.1 and the fact that ϕ has ∂-order k we have that $\partial\phi\wedge\ldots\wedge\partial^k\phi$ is a local holomorphic section of $\Lambda^k\phi^{-1}TN^{\mathbb{C}}$ which does not vanish identically and hence has only isolated zeroes. Let z_o be such a zero, then on a neighbourhood of z_o,

$$\partial\phi\wedge\ldots\wedge\partial^k\phi = (z-z_o)^m\omega$$

where ω is a non-zero decomposible multivector, which thus defines a local rank k subbundle of $\phi^{-1}TN^{\mathbb{C}}$ which coincides with W_ϕ^k off z_o. Thus we have defined a global rank k subbundle $\tilde{\psi}$ of $\phi^{-1}TN^{\mathbb{C}}$ which coincides with W_ϕ^k off an isolated set of points.

Since W_ϕ^k is isotropic by the isotropy of ϕ, $\tilde{\psi}$ is also isotropic and thus defines a map ψ: $M^2 \to F_k(N,h)$ which we claim is horizontal and f-holomorphic.

Firstly $\partial\phi \in W_\phi^k$ and thus $\partial\phi \in \tilde{\psi}$. Now let $\delta \in C^\infty(\tilde{\psi})$. Then on a dense set, we have

$$\delta = \sum_{i=1}^{k} \lambda_i \partial^i \phi$$

and thus

$$\nabla_X \delta = \sum_{i=1}^{k} (X\lambda_i . \partial^i\phi + \lambda_i \nabla_X \partial^i \phi).$$

By Lemma 2.1

$$\nabla_{\frac{\partial}{\partial \bar{z}}} \partial^i \phi \in W_\phi^{i-1}$$

while the fact that ϕ has ∂-order k ensures that $\partial^\alpha \phi \in W_\phi^k$ for all α. Thus $\nabla_X \delta \in \tilde{\psi}$ on a dense set and hence

$$\nabla_X C^\infty(\tilde{\psi}) \subset C^\infty(\tilde{\psi})$$

for all $X \in TM$. An appeal to theorem 3.1 of Chapter 6 now completes the proof. □

Remarks 1. The restriction on the curvature of N is perhaps not surprising since Rawnsley [59] has shown that it is only under these conditions that the F_2 structure is reasonably well-behaved (i.e. F_2 satisfies condition A). In the special case that dim $N = 4$, Eells and Salamon [26] have shown that we may relax the curvature conditions to the demand that N be Einstein and either self-dual or anti-self-dual. We shall discuss a possible extension of the theorem below.

2. An examination of the proof, shows that for $\alpha < k$, W_ϕ^α may be extended to give an isotropic subbundle of $\phi^{-1}TN^{\mathbb{C}}$ and thus a map $\psi_\alpha: M^2 \to F_\alpha(N,h)$ which is F_2-holomorphic by Lemma 2.1 but non-horizontal. In particular, in view of the fact that F_2 satisfies condition A, each ψ_α is harmonic with respect to a suitable metric on $F_\alpha(N,h)$.

3. Rawnsley [59] has proved a similar theorem for complex isotropic harmonic maps of surfaces into spaces of constant holomorphic curvature. Letting $\partial^{(1,0)}\phi$ denote the projection of $\partial\phi$ onto $T^{(1,0)}N$ and k the largest integer for which

$$\partial\phi^{(1,0)}\wedge\ldots\wedge\nabla^{k-1}_{\frac{\partial}{\partial z}}\partial^{(1,0)}\phi \not\equiv 0 \quad ,$$

he constructs a horizontal holomorphic map $\psi: M^2 \to \mathbb{G}_k(T^{(1,0)}N)$ given by

$$\psi(x) = \mathrm{span}_x\{\partial\phi^{(1,0)}, \ldots, \nabla^{k-1}_{\frac{\partial}{\partial z}}\partial^{(1,0)}\phi\}$$

off a set of isolated points. As in remark 2, for $\alpha < k$ one can construct J_2 holomorphic non-horizontal maps $\psi_\alpha: M^2 \to \mathbb{G}_\alpha(T^{(1,0)}N)$ which are harmonic In particular, if $N = \mathbb{C}P^n$ we can construct in this way harmonic, non-holomorphic maps into flag manifolds and Grassmanians using the results of §5, of Chapter 6.

3. *A Bijective Correspondence*

To use theorem 2.2 to obtain a bijective correspondence between isotropic harmonic maps and f-holomorphic horizontal maps, we must characterise those f-holomorphic horizontal maps into $F_k(N,h)$ which project onto maps of ∂-order k. To do this, we follow Rawnsley and exploit the D connection on $F_k(N,h)$ mentioned in §1, Chapter 6.

Proposition 3.1 Let $\psi: M^2 \to F_k(N,h)$ be horizontal. Then $\phi = \pi_o\psi$ has ∂-order k if and only if

$$\psi_*\frac{\partial}{\partial z} \wedge \ldots \wedge \psi^{-1}D^{k-1}_{\frac{\partial}{\partial z}}\psi_*\frac{\partial}{\partial z} \not\equiv 0 .$$

Proof By proposition 1.1 of Chapter 6

$$D = \pi^{-1}\nabla - P \quad \text{on} \quad \mathcal{H}$$

where P is projection onto the vertical distribution of $F_k(N,h)$ and ∇ is, as usual, the connection on N.

Thus, for ψ horizontal

$$\psi^{-1}D = \psi^{-1}\pi^{-1}\nabla - \psi_*P = \phi^{-1}\nabla$$

and the result now follows from the isomorphism of $\psi^{-1}\mathcal{H}$ and $\phi^{-1}TN$ via π_*.

□

Call a horizontal map satisfying the above condition *non-degenerate* and then we have, putting together propositions 1.1 and 3.1 and theorem 2.2, *Theorem 3.2* Let M^2 be a connected Riemann suface and (N,h) a space form. There is a bijective correspondence between harmonic isotropic maps $\phi: M^2 \to (N,h)$ of ∂-order k and non-degenerate f-holomorphic, horizontal maps $\psi: M^2 \to F_k(N,h)$ given by

$$W_\phi^k \subset \tilde{\psi} \; .$$

4. *Extensions and Final Remarks*

An examination of the proof of theorem 2.2 reveals that the curvature hypothesis on N is only used in Lemma 2.1 and thus following Rawnsley we may consider *curvature isotropic* maps defined as follows:

Definition 4.1 Let $\phi: M^2 \to (N,h)$ be a smooth map and let R denote the curvature tensor of (N,h). ϕ is said to be curvature isotropic if

i) ϕ is isotropic

ii) $\phi^*R(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})\, W_\phi^\alpha \subset W_\phi^\alpha$ for any isothermal co-ordinate z.

Then repeating the proof of theorem 2.2 we have

Theorem 4.2 Let $\phi: M^2 \to (N,h)$ be a harmonic, curvature-isotropic map of ∂-order k, then there is a unique horizontal, f-holomorphic map $\psi: M^2 \to F_k(N,h)$ such that $\pi_o\psi = \phi$ characterised by

$$W_\phi^k \subset \tilde{\psi} \; .$$

Definition 4.2 Let $\phi: M^2 \to F_k(N,h)$ be a smooth map, then ψ is *D-curvature isotropic* if

$$\psi^* R^D\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) \mathrm{span}\{\psi_* \frac{\partial}{\partial z}, \ldots, \psi^{-1} D^k_{\frac{\partial}{\partial z}} \psi_* \frac{\partial}{\partial z}\} \subset \mathrm{span}\{\psi_* \frac{\partial}{\partial z}, \ldots, \psi^{-1} D^k_{\frac{\partial}{\partial z}} \psi_* \frac{\partial}{\partial z}\}$$

for all k.

Arguing as in proposition 3.1 we see that if ψ is horizontal and f-holomorphic, $\phi = \pi_o \psi$ is curvature isotropic if and only if ψ is D-curvature isotropic and so we have

Theorem 4.3 Let M^2 be a connected Riemann surface and (N,h) a Riemannian manifold. There is a bijective correspondence between harmonic curvature-isotropic maps $\phi: M^2 \to (N,h)$ of ∂-order k and non-degenerate, D-curvature isotropic horizontal f-holomorphic maps $\psi: M^2 \to F_k(N,h)$, given by

$$W^k_\phi \subset \tilde{\psi} .$$

However, there are no known examples of curvature-isotropic maps apart from the isotropic harmonic maps into space forms.

Remarks

1. If $N = S^{2n}$, theorem 3.2 reduces to a version of Calabi's famous theorem [13] since it is easily shown that $\phi: M^2 \to S^{2n}$ has ∂-order n if and only if ϕ is full (i.e. not contained in any proper totally geodesic subspace of S^{2n}) and so in this case our correspondence is between full isotropic harmonic maps into S^{2n} and holomorphic horizontal maps into $J(S^{2n}) = \frac{O(n+1)}{U(n)}$. Further in this case, the connection D

may be identified with the canonical connection on $\frac{O(n+1)}{U(n)}$ induced by the decomposition of o(n) discussed in the appendix to Chapter 6.

2. As we saw in Chapter 4, proposition 2.3, the isotropy of a harmonic map of a surface M^2 into a space form depended on the vanishing of certain holomorphic differentials on M^2. In particular, since S^2 admits no non-zero holomorphic differentials, any harmonic map of S^2 into a space form is isotropic. Thus theorem 3.2 classifies *all* harmonic maps $\phi: S^2 \to N$, for N a space form. Further, by applying corollary 1.3, we recover in this way the result of Almgren that any minimal immersion of S^2 into S^3 is totally geodesic.

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